Mathematical Reviews

Edited by

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J. V. Wehausen, *Executive Editor*

Vol. 15, No. 9 October, 1954 pp. 769-844

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MATHEMATICAL REVIEWS

Published monthly, except August, by

The American Mathematical Society, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

The American Mathematical Society
The Mathematical Association of America
The Institute of Mathematical Statistics
The Edinburgh Mathematical Society
Société Mathématique de France
Dansk Matematisk Forening
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Editorial Office

Mathematical Reviews, 80 Waterman St., Providence 6, R. I.

Subscriptions: Price $20 per year ($10 per year to members of sponsoring societies). Checks should be made payable to Mathematical Reviews. Subscriptions may be addressed to Mathematical Reviews, Lancaster, Pennsylvania, but should preferably be addressed to the American Mathematical Society, 80 Waterman St., Providence 6, R. I.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Scientific Research, Headquarters, Air Research and Development Command, U. S. Air Force. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in paragraph (d-2), section 34.40, P. L. and R. of 1948, authorized November 9, 1940.

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Mathematical Reviews

Vol. 15, No. 9 OCTOBER, 1954 Pages 769–844

HISTORY

*Kline, Morris. **Mathematics in western culture.** Oxford University Press, New York, 1953. xii+484 pp. (27 plates). $7.50.

Although the material in this book is arranged in a historical order, it is not a history of mathematics; the history (of which there is necessarily a great deal) is only incidental to the main motive, which is to discuss the cultural significance of mathematics in the development of Western civilization. The author expresses his thesis in the Preface: ". . . mathematics has determined the direction and content of much philosophic thought, has destroyed and rebuilt religious doctrines, has supplied substance to economic and political theories, has fashioned major painting, musical, architectural, and literary styles, has fathered our logic, and has furnished the best answers we have to fundamental questions about the nature of man and his universe. As the embodiment and most powerful advocate of the rational spirit, mathematics has invaded domains ruled by authority, custom, and habit, and supplanted them as the arbiter of thought and action. Finally, as an incomparably fine human achievement mathematics offers satisfactions and aesthetic values at least equal to those offered by any other branch of our culture."

Since the book is intended for non-mathematical as well as mathematical readers, the first chapter is devoted to the general nature and chief characteristics of mathematics, especially as regards method, artistic quality, symbolic aspects, position as a body of knowledge and as a living and growing entity. The general characteristics of the mathematics of the ancient Orient and Greece are then analyzed, as well as the shaping of mathematical thought by these ancient cultures and its reciprocal influence on the latter. In a chapter entitled "The birth of the mathematical spirit," the author declares, "The Greek preference for deduction was, surprisingly, a facet of the Hellenic love for beauty. . . . Beauty was an intellectual as well as an emotional experience. Indeed, the Greek sought the rational element in every emotional experience." Also, "The slave basis of classical Greek society fostered a divorce of theory from practice and the development of the speculative and abstract side of science and mathematics with a consequent neglect of experimentation and practical applications." He judges that "the contribution of the Greeks that did most to determine the character of present-day civilization was their mathematics." The development of Euclid's geometry, the founding of the Museum and the fusing of practice with mathematics at Alexandria, and the work of Eratosthenes, Archimedes, Hipparchus and Ptolemy are all discussed as part of their cultural milieu.

The period from 500 to 1400 A.D. is given the title "Interlude," and touches on the influences operating between theology and mathematics during the period; notice is taken of the progress in mathematics made by the Hindus and Arabs, of course. With the re-introduction of Greek works into Europe by the Arabs, rise of the merchant and trading class, exploration, etc., the idea grew that man might master nature (F. Bacon, Descartes) through the medium of mathematically expressed laws. "Many people credit the rise of modern science to the introduction of experimentation on a large scale and believe that mathematics served only occasionally as a handy tool. The true situation . . . was actually quite the reverse." "For Galileo, Descartes, Huygens, and Newton, the deductive, mathematical part of the scientific enterprise always loomed larger than experimentation." Copernicus and Kepler sought and became enchanted by the mathematical harmony of their astronomical laws. "The scientific Declaration of Independence is a collection of mathematical theorems."

Two chapters are devoted to the development of rules of perspective in painting and its influence on the creation of projective geometry. There follows a series of chapters on the growth of the "scientific method" (Descartes, Fermat, Galileo, Newton) and the effects on philosophical thought, religion, literature, aesthetics, etc.; and two chapters describing in a very effective and elementary manner the mathematical analysis of wave motion and the resulting discoveries of Maxwell. An entertaining account is given of the 17th and 18th century attempts to emulate the success of the mathematical method in the physical world by setting up a "Science of human nature", a series of endeavors possibly little known to the average mathematician, followed by a description of the rise of the more successful methods of statistics and probability. Concluding chapters are devoted to Cantor's theory of the infinite, non-euclidean geometry, and relativity, and their impact on modern thought; and, finally, a chapter on the flourishing state of modern mathematics.

The typography is excellent and in addition to the many (88) figures in the text a series of twenty-seven plates is assembled in readily accessible form after the table of contents.

In the reviewer's opinion, the author has well sustained his thesis, quoted above, and has produced a work that no mathematician, and no scientist, can afford to ignore; and a similar remark holds for the layman who wishes to be well-informed on the development of Western culture.

R. L. Wilder (Ann Arbor, Mich.).

*Whittaker, Edmund. **A history of the theories of aether and electricity. Vol. II. The modern theories, 1900–1926.** Philosophical Library, New York, N. Y., 1954. xi+319 pp. $8.75.

This volume contains a story starting with Becquerel's great discovery and ending with the introduction of exchange interaction by Heisenberg. The story is told in nine chapters, as follows: The age of Rutherford, The relativity of Poincaré and Lorentz, The beginnings of quantum theory, Spectroscopy in the older quantum theory, Gravitation, radiation and atoms in the older quantum theory, Mag-

netism and electromagnetism, 1900–1926, The discovery of matrix-mechanics, The discovery of wave-mechanics.

One can marvel at the erudition of the author, at the great number of papers quoted, and the extensive knowledge of interesting details. Yet the proportion seems somehow distorted. Thus, for example, in Chapter II, The relativity theory of Poincaré and Lorentz, the name of Einstein is mentioned only three times. The first time it is introduced in the following way: "In the autumn of the same year, in the same volume of the Annalen der Physik as his paper on the Brownian motion, Einstein published a paper which set forth the relativity theory of Poincaré and Lorentz with some amplifications, and which attracted much attention. He asserted as a fundamental principle the constancy of the velocity of light, i.e. that the velocity of light in vacuo is the same in all systems of reference which are moving relative to each other: an assertion which at the time was widely accepted, but has been severely criticised by later writers. In this paper Einstein gave the modifications which must now be introduced into the formulae for aberration and the Doppler effect."

Thus the emphasis here and elsewhere is not the accepted one. However, those interested in the history of the development of theoretical physics in this quarter century will find it a fascinating book. *L. Infeld* (Warsaw).

Itard, Jean. Quelques remarques historiques sur l'axiomatique du concept de grandeur. Revue Sci. **91**, 3–14 (1953).

de Dainville, François. L'enseignement des mathématiques dans les Collèges Jésuites de France du XVI^e au XVIII^e siècle. Rev. Hist. Sci. Appl. **7**, 6–21 (1954).

Novák, J., Vyčichlo, F., and Zelinka, R. Sixtieth birthday of academician Eduard Čech. Čehoslovack. Mat. Ž. **3(78)**, 183–198 (1 plate) (1953). (Russian)

A list of Čech's published mathematical work is included.

★Eulerus, Leonhardus. Opera omnia. Series prima. Opera mathematica. Vol. XXVI. Commentationes geometricae. Vol. primum. Edidit Andreas Speiser. Societas Scientiarum Naturalium Helveticae, Lausanne, 1953. xxxviii+362 pp.

This volume begins with an illuminating 30-page Introduction by A. Speiser, who has edited the work with great care, correcting many small errors. The first paper, on the areas of lunes in a plane, is reminiscent of Hippocrates of Chios and Leonardo da Vinci. Another elementary paper is on the quadrisection of a scalene triangle by two perpendicular cuts. "Solutio facilis problematum quorundam geometricorum difficillimorum" contains many properties of the principal centers of a triangle, such as the condition for the in-center to lie on the Euler line. "Elementa doctrinae solidorum" is of special interest because it contains the polyhedral formula $H+S=A+2$ and many consequences, such as the fact that no polyhedron can have just seven edges. "De corporibus regularibus . . ." employs spherical trigonometry to obtain the metrical properties of the Platonic solids inscribed in a sphere of radius r. In other papers, the classical formulas of spherical trigonometry are elegantly derived, and it is interesting to observe that most of them are written exactly as we would write them to-day. One particularly remarkable theorem is that the cevians Aa, Bb, Cc, drawn through a point O inside a spherical triangle ABC, satisfy

$$\alpha\beta\gamma=\alpha+\beta+\gamma+2,$$

where

$$\alpha=\tan AO/\tan Oa,\quad \beta=\tan BO/\tan Ob,\quad \gamma=\tan CO/\tan Oc.$$

The only articles not in Latin are two in French on the number of intersections of two plane algebraic curves, and one in German (by Leonhard's son Albrecht) on the dissection of various plane figures by parallel lines into several pieces of equal area. *H. S. M. Coxeter* (Toronto, Ont.).

Drobot, S. L'oeuvre scientifique de M. T. Huber (4. I. 1872–9. XII. 1950). Colloquium Math. **3**, 63–72 (1 plate) (1954).

Dugas, René. Sur le cartésianisme de Huygens. Rev. Hist. Sci. Appl. **7**, 22–33 (1954).

Montgomery, Deane, and Veblen, Oswald. Obituary: Nels Johann Lennes. Bull. Amer. Math. Soc. **60**, 264–265 (1954).

Frank, Lyudvik. On the thirtieth anniversary of the death of the Czech mathematician Mathias Lerch. Čehoslovack. Mat. Ž. **3(78)**, 109–110 (1953). (Russian)

Škrašek, Iosef. List of works of Prof. Mathias Lerch. Čehoslovack. Mat. Ž. **3(78)**, 111–122 (1953). (Russian)

Linnik, Yu. V., and Šanin, N. A. Andreĭ Andreevič Markov (on his fiftieth birthday). Uspehi Matem. Nauk (N.S.) **9**, no. 1(59), 145–149 (1 plate) (1954). (Russian)

Žautikov, O. A. Konstantin Petrovič Persidskiĭ (on his fiftieth birthday). Uspehi Matem. Nauk (N.S.) **9**, no. 1(59), 151–154 (1 plate) (1954). (Russian)

★Ruffini, Paolo. Opere matematiche. Tomo Secondo. Pubblicate sotto gli auspici dell' Unione Matematica Italiana a cura del Prof. Dr. Ettore Bortolotti con il contributo del Consiglio Nazionale delle Richerche. Edizioni Cremonese della Casa Editrice Perella, Roma, 1953. xviii+509 pp.

Vol. I of Ruffini's mathematical works was published in 1915 by the Circolo Matematico di Palermo. Vol. II was first published in 1943, but all except three or four copies were destroyed by bombs. The present edition is reproduced by photographic methods from one of these surviving copies.

★Ruffini, Paolo. Opere matematiche. Tomo Terzo. Carteggio matematico. Pubblicate sotto gli auspici dell'Unione Matematica Italiana a cura del Prof. Dr. Ettore Bortolotti con il contributo del Consiglio Nazionale delle Richerche. Edizioni Cremonese della Casa Editrice Perrella, Roma, 1954. xvii+254 pp. 3000 Lire.

Szegő, Gabor. Obituary: Otto Szász. Bull. Amer. Math. Soc. **60**, 261–263 (1954).

Hofmann, Jos. E. François Viète und die Archimedische Spirale. Arch. Math. **5**, 138–147 (1954).

This short account of Vieta (1540–1603) contains a remarkable amount of information and of references to original works and commentaries that are not easily come by. Emphasis is placed on the strategic position which Vieta

occupied, linking the Aristotelian view of mathematical scope and proof with the revolutionary change that came in the 17th century. One hitherto unnoticed method of Vieta is discussed, which is found in his Variorum de rebus mathematicis responsorum liber VIII [Tours, 1593], a work of rich content: the early seven books are lost. Vieta investigated the spiral of Archimedes $r=a\phi$ (in polar coordinates r, ϕ), and in particular shews that if P_+, P, P_- are three points on the curve, at which $\phi=\frac{1}{2}\pi+\epsilon$, $\frac{1}{2}\pi$, $\frac{1}{2}\pi-\epsilon$, then the external bisector of the angle P of the triangle P_+PP_- is a close approximation to the tangent at P: and further that as ϵ takes values 1/20, 1/40, 1/80 of $\frac{1}{2}\pi$ the approximation improves. Vieta gives only a numerical illustration of these statements, but the author works the processes out in detail, and proves them. Such a procedure by Vieta is seen to be of great importance, for it is the earliest known example of finding the tangent to a curve by a direct process of approximation. *H. W. Turnbull.*

FOUNDATIONS

Šanin, N. A. On imbeddings of the classical logico-arithmetical calculus into the constructive logico-arithmetical calculus. Doklady Akad. Nauk SSSR (N.S.) **94**, 193–196 (1954). (Russian)

This is a continuation of a previous paper by the same author [same Doklady (N.S.) **93**, 779–782 (1953); these Rev. **15**, 593]. The subject is mappings of the predicate calculus with arithmetic which carry true formulas of the classical calculus into true formulas of the intuitionistic calculus. Such mappings have been constructed by Kolmogorov and Gödel. If α (i.e. the mapping $p\rightarrow\alpha p$) is such a mapping, the author calls α regular if and only if the formula $\alpha p\supset p$ is valid in the intuitionistic calculus. The author gives some examples of regular mappings, and states their relationships to one another and to the irregular mappings of Kolmogorov and Gödel. *H. B. Curry.*

Rose, Alan. A formalisation of the 2-valued propositional calculus with self-dual primitives. Math. Ann. **127**, 255–257 (1954).

Post [The two-valued iterative systems of mathematical logic, Princeton, 1941; these Rev. **2**, 337] showed that conditioned disjunction $[X, Y, Z]$, namely $(X\&Y)\vee(Z\&\bar{Y})$, together with the non-logical constants t and f form a complete, independent set of primitives for the two-valued propositional calculus. Church [Portugaliae Math. **7**, 87–90 (1948); these Rev. **10**, 421] observed that these primitives are self-dual, the dual of an expression being obtained by writing its symbols in reverse order and interchanging t and f; a self-dual definition of the negation of A being $[f, A, t]$. The author establishes these results on the basis of a neat, weakly complete formalization of Church's system using one axiom, namely t, and three independent rules of inference. *G. Kreisel* (Reading).

Péter, Rózsa. Rekursive Definitionen, wobei frühere Funktionswerte von variabler Anzahl verwendet werden. Publ. Math. Debrecen **3** (1953), 33–70 (1954).

Let α, F be primitive recursive functions (prf) and let

$$(1)\quad\begin{cases}\phi(n_1,\cdots,n_k,a)=\alpha(n_1,\cdots,n_k,a) \text{ if } n_1\cdot\cdots\cdot n_k=0,\\ \phi(n_1+1,\cdots,n_k+1,a)\\ \qquad=F(n_1,\cdots,n_k,a,\phi(n_1,x_{1,1},\cdots,x_{1,k}),\\ \qquad\quad\phi(n_1+1,n_2,x_{2,1},\cdots,x_{2,k-1}),\\ \qquad\quad\cdots,\phi(n_1+1,\cdots,n_{k-1}+1,n_k,x_{k,1})),\end{cases}$$

where the $x_{i,j}$ are terms. It is shown that these recursions can be reduced to substitutions and multiple recursions. Moreover, if in a recursion of the form (i) the function

$$F(n_1,\cdots,n_k,a,\xi_1(x_{1,1},\cdots,x_{1,k}),\\ \xi_2(x_{2,1},\cdots,x_{2,k-1}),\cdots,\xi_k(x_{k,1}))$$

has nested occurrences of ξ_1 only, then ϕ is a prf, but a single nested occurrence of, for example, $\xi_2(x_{2,1},\cdots,x_{2,k-1})$ may lead out of the class of prfs. Application of the results to a function introduced by Bereczki leads to a new proof that the elementary functions are a proper subclass of the prfs. There is also a new proof that the induction scheme

$$\phi(m+1,n+1,a)=\prod_{i=1}^{a-1}\max(\phi(m+1,n,i),\phi(m,i,a))$$

introduced by Skolem does not lead outside prfs.

I. Novak Gál (Ithaca, N. Y.).

Markwald, Werner. Zur Theorie der konstruktiven Wohlordnungen. Math. Ann. **127**, 135–149 (1954).

The author considers recursively enumerable (r.e.) relations, specified by the Gödel numbers of their defining equations; $\bar{x}$ denotes the ordinal of the well-ordering relation x. His main result is this: given any predicate A of elementary number theory (e.n.) he exhibits two r.e., actually decidable, well orderings x_a, y_a, both of order $<\omega^\omega$, such that $A(x_a, y_a)\leftrightarrow\bar{x}_a\neq\bar{y}_a$; the proof uses (i) the lemma: for every $A(x)$ of e. n. there is a recursive predicate P such that $A(x)\leftrightarrow U_{x_1}\cdots U_{x_n}P(x_1\cdots x_n, x)$, where U_x means "for infinitely many x", and (ii) the observation that the lexicographic ordering $<_P$ of those $\langle x_1\cdots x_nx_{n+1}\rangle$ which satisfy $P(x_1\cdots x_nx_{n+1})$ has the ordinal ω^{n+1} if and only if $U_{x_1}\cdots U_{x_{n+1}}P(x_1\cdots x_{n+1})$. The result shows that "isomorphism" between decidable well orderings (even if $<\omega^\omega$) is not definable in e.n. It applies not only to the class W of all r.e. well orderings, but to every class (M, Θ, H) (of symbols for ordinals) which can be used to present r.e. well orderings as follows: (i) $\Theta(x)$, $H(x)$ are partially recursive, (ii) $x\in M\leftrightarrow\Theta(x)\in(M, \Theta, H)$ and $(M, \Theta, H)\subset W$, (iii) for every $x\in W$ there is a $y\in M$ such that $\overline{\Theta(y)}=\bar{x}$, (iv) if $x\in W$ then $\overline{\Theta[H(x)]}=\theta(\bar{x})$, θ strictly increasing; in particular, the Church-Kleene notation for constructive ordinals [Fund. Math. **28**, 11–21 (1937)] satisfies these conditions. The author raises the question whether "well ordering" can be defined in e.n., and shows that it cannot be defined by means of the prefix $\exists\forall\exists$, thereby settling a point raised by Kleene [Introduction to metamathematics, Van Nostrand, New York, 1952, p. 527; these Rev. **14**, 525]. The author's work shows that the property of isomorphism between decidable well orderings has no base in e.n. [cf. the reviewer's paper in British J. Philos. Sci. **4**, 107–129 (1953), 357 (1954); these Rev. **15**, 670]; for the ordering got by joining the orderings $<_P$ for all P with $n=1$, then $n=2$, etc., is decidable and of order ω^ω, but there is no function of e.n. which maps this ordering isomorphically on one of the standard orderings of order ω^ω. *G. Kreisel.*

Halmos, Paul R. Polyadic Boolean algebras. Proc. Nat. Acad. Sci. U. S. A. **40**, 296–301 (1954).

Für die Aufgabe eine abstrakte Struktur zu finden, die sich zum Prädikatenkalkül so verhält, wie der Boolesche Verband zum Aussagenkalkül, wird hier eine neue Lösung

vorgeschlagen. Zunächst werden als "functional algebras" Boolesche Verbände von Abbildungen einer Menge X in einen Booleschen Verband eingeführt, die mit einer Abbildung p stets auch die konstante Abbildung $\exists p$ enthalten mit (1) $p(x) \leqq \exists p$ für alle x, (2) $p(x) = \exists p$ für ein x. Wird X durch eine Potenz X^I (mit beliebiger Indexmenge) ersetzt, so entstehen weitere "functional algebras". Verf. gibt ein Axiomensystem an, so dass jede "polyadische Algebra", d.h. jedes Modell des Systems, isomorph in einer "functional algebra" enthalten ist. Der Gödelsche Vollständigkeitssatz ergibt sich dann als Spezialfall des abstrakten Satzes: jede polyadische Algebra ist semi-einfach. Die Beweise sind nicht ausgeführt. *P. Lorenzen* (Bonn).

Kleene, S. C., and Post, Emil L. The upper semi-lattice of degrees of recursive unsolvability. Ann. of Math. (2) **59**, 379–407 (1954).

Nach Kleene, IM [Introduction to metamathematics, Van Nostrand, New York, 1953, S. 275; diese Rev. **14**, 525], ist die Relation, "A ist rekursiv in B" für Relationen (oder 2-wertige Funktionen) von natürlichen Zahlen reflexiv und transitiv. Die Relation, "A ist rekursiv in B und umgekehrt", ist daher eine Gleichheitsrelation. Durch Abstraktion bzgl. dieser Gleichheitsrelation stellt jede Relation einen "Unlösbarkeitsgrad" dar. Es wird $a \leqq b$ gesetzt, wenn für die Relationen A und B mit den Graden a und b gilt, dass A rekursiv in B ist. Die rekursiven Relationen haben den Grad 0 und $0 \leqq a$ für alle Grade a. Die Grade bilden einen $\vee$-Halbverband, denn zu zwei Relationen A und B gibt es stets eine Relation D, so dass (1) A und B rekursiv in D sind, (2) D rekursiv in A, B ist. Der Grad von D ist daher das k.g.V. $a \vee b$ der Grade a und b von A und B. Für jede Relation A gibt es in der Klasse der Formeln $(Ex)R(y, x)$ mit in A rekursivem R eine Formel $C^A(y)$, so dass jede Formel der Klasse zu einer Formel $C^A(\phi(y))$ mit rekursivem ϕ äquivalent ist. Der Grad von $\hat{y}C^A(y)$ hängt nur vom Grad a von A ab, und wird mit a' bezeichnet. Wird für $C^A(y)$ stets $(Ex)T_1{}^A(y, y, x)$ [vgl. IM, S. 291] gewählt, so liefert die Definition $L_0{}^A(y) = y \varepsilon A$, $L^A{}_{k+1}(y) = C^{L_k^A}(y)$ eine Relativen $\hat{k}\hat{y}L_k{}^A(y)$, deren Grad $a^{(\omega)}$ ebenfalls nur von a abhängt. Es gilt $a < a' < a'' < \cdots < a^{(n)} < \cdots < a^{(\omega)}$. Über den Halbverband der Grade beweisen die Verff. interessante Struktursätze, z.B.: (1) Zwischen a und a' gibt es eine in sich dichte Kette von Graden; (2) zwischen a und a' gibt es eine unendliche unabhängige Menge von Graden; (3) der Halbverband ist nicht ω-vollständig und ist kein Verband; (4) es gibt Grade b mit $a^{(n)} < b$ für alle n und $b < a^{(\omega)}$.

P. Lorenzen (Bonn).

van der Waerden, B. L. Einfall und Überlegung in der Mathematik. III. Elemente der Math. **9**, 49–56 (1954).

For parts I and II see same journal **8**, 121–129 (1953); **9**, 1–9 (1954); these Rev. **15**, 279.

Kotsakis, D. Mathematical simplicity and elegancy in natural research. Bull. Soc. Math. Grèce **28**, 51–62 (1954). (Greek. English summary)

ALGEBRA

Eaves, J. C. A note on sets of matrices simultaneously reducible to the triangular skeleton. J. Math. Physics **32**, 302–306 (1954).

The author studies the reduction of matrices to triangular form or to block matrices in which (1) the diagonal blocks are square matrices of triangular form and (2) the blocks off the main diagonal can be bordered by columns of zeros on the right or by rows of zeros below to become triangular and (3) the so bordered blocks below the main diagonal have only zeros in their main diagonals.

O. Taussky-Todd (Washington, D. C.).

Egerváry, E. On the hermitian normalform of a matrix and Sylvester's law of nullity. Publ. Math. Debrecen **3** (1953), 144–149 (1954).

A short constructive derivation is given of Hermite's canonical form WA for a matrix A under premultiplication by a non-singular matrix W. This is used to prove Sylvester's law of nullity. It is pointed out that the system $[W]Ax=0$ of linear homogeneous equations can be conveniently solved by making use of the fact that $WA(E-WA)$ is the 0-matrix.

J. L. Brenner (Aberdeen, Md.).

Parker, W. V. A note on normal matrices. Amer. Math. Monthly **61**, 330–331 (1954).

Let A be the partitioned matrix $(A_{\alpha\beta})$,

$$D = \text{diag } (A_{11}, A_{22}, \cdots).$$

Hypotheses: $\det (A-\lambda I) = \det (D-\lambda I)$; A normal. Conclusion: $A=D$. A short proof is the following: Choose unitary $U = \text{diag } (U_1, U_2, \cdots)$ so that

$$UDU^* = \text{diag } (U_1A_{11}U_1^*, U_2A_{22}U_2^*, \cdots)$$

is triangular. From the relations

$$\det (UAU^*-\lambda I) = \det (A-\lambda I) = \det (D-\lambda I) = \det (UDU^*-\lambda I),$$

it follows that the characteristic roots of UAU^* are its diagonal elements. By a previous theorem of the author [Duke Math. J. **15**, 439–442 (1948), Theorem 3; these Rev. **10**, 4], UAU^* is diagonal; thus UAU^* is equal to UDU^*, Q.E.D. *J. L. Brenner* (Aberdeen, Md.).

Parker, W. V., and Rutledge, W. A. Equivalence of matrices over a polynomial domain. J. London Math. Soc. **29**, 172–177 (1954).

The following theorem is proved: Let $M=(m_{ij})$ be a $k \times k$ block matrix, where m_{ii} is the companion matrix of a polynomial $f_i(x)$ of degree n_i with highest coefficient 1 and $m_{ij} (i \neq j)$ has as last row the coefficients of a polynomial $f_{ij}(x)$ of degree $n_j - 1$ and all other elements zero. Let $M(x) = (m_{ij}(x))$ be a $k \times k$ matrix, where $m_{ii}(x) = -f_i(x)$ and $m_{ij}(x) = f_{ij}(x)$, $i \neq j$. Let the coefficients of f_i and f_{ij} come from a field F. Then $M-xI$ is equivalent in $F[x]$ to

$$\begin{pmatrix} I_{n-k} & 0 \\ 0 & M(x) \end{pmatrix}.$$

Several applications are given: one of them is an alternative proof of a theorem of M. F. Smiley [Amer. Math. Monthly **56**, 542–544 (1949); these Rev. **11**, 307]; another is a pure matrix proof of a theorem on the elementary divisors of AB and BA by Flanders [Proc. Amer. Math. Soc. **2**, 871–874 (1951); these Rev. **13**, 425] who had used homomorphisms of finite vector spaces. *O. Taussky-Todd.*

Taussky, Olga. Characteristic roots of quaternion matrices. Arch. Math. **5**, 99–101 (1954).

Among other theorems, it is proved that the field of values of a quaternion matrix contains with α also $\rho^{-1}\alpha\rho$.
J. L. Brenner (Aberdeen, Md.).

Abstract Algebra

Higman, Graham, and Stone, A. H. On inverse systems with trivial limits. J. London Math. Soc. **29**, 233–236 (1954).

As a special case of a theorem of Henkin [Proc. Amer. Math. Soc. **1**, 224–225 (1950); these Rev. **11**, 675] one can conclude that there is an inverse system of sets $\{S_\alpha|\alpha$ a countable ordinal$\}$ with onto mappings, which has a void limit. The present paper starts with another example of such an inverse system (in which each S_α is countable). If V_α is the set of formal linear combinations of the elements of S_α (coefficients in an arbitrary field, for example) then $\{V_\alpha\}$ is naturally made into an inverse system of groups with onto-homomorphisms. The principal result is that this inverse system has limit consisting of zero alone. An inverse system of rings with zero limit can also be provided by using the preceding example and defining $V_\alpha^2=0$, or by using for V_α formal polynomials in the elements of S_α instead of formal linear combinations. This answers a question of the reviewer's [Duke Math. J. **18**, 431–442 (1951), p. 432; these Rev. **12**, 795]. *D. Zelinsky* (Evanston, Ill.).

Szász, G. Über die Unabhängigkeit der Assoziativitätsbedingungen kommutativer multiplikativer Strukturen. Acta Sci. Math. Szeged **15**, 130–142 (1954).

In a previous paper [same Acta **15**, 20–28 (1953); these Rev. **15**, 95] the author has shown that the set of all associativity conditions in a multiplicative system of order $\nu>3$ constitute an independent set of axioms and has determined independent sets of associativity conditions for the cases $\nu=2$ and $\nu=3$. The same problem is solved here for commutative multiplicative systems and the author constructs, for any such system, independent sets of associativity conditions whose validity ensures the associativity of the entire system. For ν finite and greater than 3 the number of such independent conditions is $\frac{1}{6}(\nu^3-\nu)$.
D. C. Murdoch (Vancouver, B. C.).

Dubreil-Jacotin, M.-L., et Croisot, R. Sur le calcul des $\mathfrak{F}$-idéaux d'une algèbre. Revue Sci. **91**, 15–18 (1953).

Given an algebra A and the lattice $\mathcal{S}(A)$ of subsets of A, each operation $a_1\phi a_2$ of A defines an operation $S_1\phi S_2$ in $\mathcal{S}(A)$ in the usual manner. Let $\mathfrak{F}$ be a family of subsets of A such that $A\in\mathfrak{F}$ and $\mathfrak{F}$ is closed under unlimited intersection. Then the members of $\mathfrak{F}$ are called the $\mathfrak{F}$-ideals of A. For $F_1,F_2\in\mathfrak{F}$, define $F_1\phi^*F_2=(F_1\phi F_2)^*=\mathfrak{F}(F_1\phi F_2)$ where $\mathfrak{F}(X)$ denotes the intersection of the elements of $\mathfrak{F}$ containing X. If $\cup$ and $\cap$ denote set union and intersection of subsets of A, then the lattice operations $\vee$ and $\wedge$ in $\mathfrak{F}$ are $\cup^*$ and $\cap^*=\cap$. Commutativity of ϕ in $\mathcal{S}(A)$ implies commutativity of ϕ^*, but not so for associativity and distributivity. Conditions are found for ϕ^* to be identical with $\vee$ or $\wedge$, and for star operations to be associative or distributive. For example, $F_1\phi^*F_2=F_1\vee F_2$ for all $F_1,F_2\in\mathfrak{F}$ if and only if (1) $F\in\mathfrak{F}$ implies $F\phi F\subset F$, and (2) $F\in\mathfrak{F}$ and $F_1\phi F_2\subset F$ imply $F_1\cup F_2\subset F$. If $a\phi b\in F\in\mathfrak{F}$ and $a\in F$ imply $b\in F$, and if $F_1,F_2\in\mathfrak{F}$ implies $F_1\phi F_2=F_1\phi^*F_2=F_1\vee F_2$, then the lattice $\mathfrak{F}$ is modular. *P. M. Whitman.*

Steinfeld, O. Remark on a paper of N. H. McCoy. Publ. Math. Debrecen **3** (1953), 171–173 (1954).

The author proves that an ideal I of a ring A is prime (completely prime) if and only if $RL\subseteq I$ ($LR\subseteq I$) implies $R\subseteq I$ or $L\subseteq I$, R an r-ideal and L an l-ideal of A. Other conditions that an ideal be prime were given by McCoy [Amer. J. Math. **71**, 823–833 (1949); these Rev. **11**, 311].
R. E. Johnson (Northampton, Mass.).

***Chatelet, Albert. Arithmétique et algèbre modernes. Tome I. Notions fondamentales—Groupes.** Presses Universitaires de France, Paris, 1954. 276 pp. 1200 francs.

This book is the first of a projected series of three on some of the basic concepts and algebraic systems of modern algebra. The first chapter has a brief introduction to logic, which is followed by sections on algebras of sets, lattices, mappings, and operations. The second and final chapter of this volume is on groups, with sections on subgroups, transformation groups, lattices of subgroups, Abelian groups, arithmetic functions on the semigroup of positive integers, and subgroups of finite groups. *R. E. Johnson.*

Almeida Costa, A. Three lectures on the general theory of rings. II. Anais Fac. Ci. Porto **36**, 169–200 (1952) = Centro Estudos Mat. Fac. Ci. Porto. Publ. no. 30, 36 pp. (1952). (Portuguese)

Almeida Costa, A. Three lectures on the general theory of rings. III. Anais Fac. Ci. Porto **36**, 221–247 (1952) = Centro Estudos Mat. Fac. Ci. Porto. Publ. no. 31, 31 pp. (1952). (Portuguese)

[For the first lecture see same Anais **36**, 65–83 (1952); these Rev. **13**, 902.] The topics covered in the second expository lecture are: primitive rings, density theorem, ideals in primitive rings, dual vector spaces. The third lecture: subdirect sums, subdirectly irreducible commutative rings, semi-simple rings, the Jacobson radical.
I. Kaplansky (Chicago, Ill.).

Osima, Masaru. Notes on basic rings. II. Math. J. Okayama Univ. **3**, 121–127 (1954).

[For part I see same J. **2**, 103–110 (1953); these Rev. **14**, 941.] Let R be a ring and V an R-module. It is assumed that V has a unique expression as a direct sum of a finite number of indecomposable modules. By taking one representative from each group of isomorphic ones, a basic submodule V_0 is defined. Some questions concerning V are reducible to V_0: for instance, the double commutator theorem holds for V if and only if it holds for V_0. In the second section of the paper the author returns to the topic of basic rings for the case of a finite-dimensional algebra, and studies their behavior under Kronecker products and scalar extensions. *I. Kaplansky* (Chicago, Ill.).

Gröbner, Wolfgang. Über das Verhalten der Hilbertfunktion eines H-Ideals bei rationalen Transformationen. Arch. Math. **5**, 1–3 (1954).

Let A_x be an H-ideal in $R_x=K[x_0,x_1,\cdots,x_m]$, where K is a field and $x_0,x_1,\cdots,x_m$ are indeterminates, let $\varphi_0(x),\varphi_1(x),\cdots,\varphi_n(x)$ be forms of the same degree β, and let

$A_y=[F(y)\in R_y=K[y_0,y_1,\cdots,y_n];\ F(\varphi_0,\varphi_1,\cdots,\varphi_n)\in A_x]$,

where F is a form. If $U=\sum R_s\varphi_j$, then

$$H(t, A_y)=H(\beta t, A_s)-H(\beta t, A_s+U^t),$$

where H denotes the Hilbert characteristic function. This formula is used to compute $H(t, A_y)$ when U is restricted in various ways. *H. T. Muhly* (Iowa City, Iowa).

Curtis, Charles W. The structure of non-semisimple algebras. Duke Math. J. 21, 79–85 (1954).

The author's main result is as follows. Let A be an algebra over a field K and let N denote the radical (in the sense of Jacobson). Suppose further that A is complete in the N-adic topology, i.e., the topology defined by taking the cosets $x+N^i$, $i=1, 2, \cdots$, to be the neighborhoods of x. If further $\bigcap N^i=(0)$ and if finally one assumes that A/N is finite-dimensional and separable over K, then: (1) A contains a subalgebra B such that $B+N=A$, $B\cap N=(0)$; (2) if B' is a subalgebra of A such that $B'+N=A$, $B'\cap N=0$, then there exists an element $z\in N$ such that if z' is the quasi-inverse of z, the mapping $b'\rightarrow b'-zb'-b'z+zb'z'$ is an isomorphism of B' onto B. The first part of this result, a generalisation of Wedderburn's theorem, is a direct consequence of the assumed completeness, while for the proof of the second part, which is an extension of A. Malcev's result [C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 42–45 (1942); these Rev. **4**, 130] the author uses in addition a theorem due to Hochschild [Ann. of Math. (2) **46**, 58–67 (1945); these Rev. **6**, 114].

A ring S with unit is said to be a finite extension of a ring R, if R lies in the center of S, R contains the unit of S, and S is a finitely generated R-module. A ring S is called semi-local, if S satisfies the maximum condition for right ideals and S/N, where N is the radical of S, satisfies the minimum condition for right ideals. The author shows that if S is a finite extension of a semi-local ring R, then S is semi-local and $\bigcap_{i=1}^{\infty}N^i=(0)$. If, further, R is complete in the M-adic topology, where M denotes the radical of R, then S is complete in the N-adic topology. Finally, the author applies these results to show that if S is a finite extension of a complete local ring R, then under suitable further assumptions S contains a certain subring B such that $B+N=S$, $B\cap N=(0)$. He also shows that B is uniquely determined up to "inner" isomorphisms in the sense specified above, thus supplementing a result due to Azumaya [Nagoya Math. J. **2**, 119–150 (1951); these Rev. **12**, 669].

J. Levitzki (Jerusalem).

Albert, A. A. The structure of right alternative algebras. Ann. of Math. (2) 59, 408–417 (1954).

The author has previously shown [Ann. of Math. (2) **50**, 318–328 (1949); these Rev. **10**, 503] that every semi-simple right alternative algebra of characteristic 0 is alternative. In this paper the characteristic is assumed only to be $\neq 2$, and the same result is proved. For this purpose the author modifies his definition of trace-admissibility [Proc. Nat. Acad. Sci. U. S. A. **35**, 317–322 (1949); these Rev. **11**, 6] as follows. Let A be a strictly power-associative algebra A over a field F of characteristic $\neq 2$, K be the algebraic closure of F, and $\delta(x)$ be a linear function on A_K. Then $\delta(x)$ is an "admissible trace function" for A if (i) $\delta(z)=0$ for every nilpotent element z of A_K, (ii) $\delta(u)\neq 0$ for every primitive idempotent u of A_K, (iii) $\delta(xy)=\delta(yx)$, and (iv) $\delta[x(yz)]=\delta[(xy)z]$ for every x, y, z of A.

The radical (maximal nilideal) N of an algebra A with admissible trace function $\delta(x)$ is the set of all x in A such that $\delta(xy)=0$ for every y in A, and is identical with the radical of the attached commutative algebra $A^{(+)}$. If A is right alternative, then $A_K^{(+)}$ is a special Jordan algebra with radical R, and the author defines an admissible trace function for A by construction in terms of natural trace functions for the simple components of $A_K^{(+)}/R$ (four types of central simple special Jordan algebras). From the fact that A has an admissible trace function, it follows readily that A is alternative if $N=0$. *R. D. Schafer.*

Jacobson, N. A Kronecker factorization theorem for Cayley algebras and the exceptional simple Jordan algebra. Amer. J. Math. 76, 447–452 (1954).

If an algebra $\mathfrak{B}$ with an identity 1 has a finite-dimensional central simple subalgebra $\mathfrak{A}$ containing 1, then $\mathfrak{B}$ is a Kronecker product $\mathfrak{B}=\mathfrak{A}\otimes\mathfrak{U}$ provided that either (I) $\mathfrak{B}$ is associative, or (II) $\mathfrak{B}$ is alternative and $\mathfrak{A}$ a Cayley algebra, or (III) $\mathfrak{B}$ is a Jordan algebra and $\mathfrak{A}$ an exceptional simple Jordan algebra. Of these, (I) is classical, due to Wedderburn. Kaplansky [Portugaliae Math. **10**, 37–50 (1951); these Rev. **13**, 8] stated (II), observing that Albert [Canadian J. Math. **4**, 129–135 (1952); these Rev. **14**, 11] gave all the arguments required to prove it. The author gives a new proof of (II) and proves the new result (III). Extensions of (II), (III) are given, dropping the assumption that the identity of $\mathfrak{A}$ is an identity in $\mathfrak{B}$. *I. M. H. Etherington.*

Schafer, R. D. On the algebras formed by the Cayley-Dickson process. Amer. J. Math. 76, 435–446 (1954).

The 8-dimensional Cayley-Dickson algebras, denoted here $\mathfrak{C}=\mathfrak{A}_3$, were constructed by Dickson from the 4-dimensional algebras $\mathfrak{Q}=\mathfrak{A}_2$ of generalized quaternions. As shown by Albert [Ann. of Math. (2) **43**, 161–177 (1942); these Rev. **3**, 261], the process can be iterated to yield a class of central simple algebras $\mathfrak{A}_t$ of dimension 2^t $(t=2, 3, 4, \cdots)$ over an arbitrary field F. The algebras $\mathfrak{Q}$ are associative, $\mathfrak{C}$ non-associative but alternative, $\mathfrak{A}_t$ $(t>3)$ non-alternative but, as the author shows, flexible $(xy\cdot x=x\cdot yx)$. Other properties of the $\mathfrak{A}_t$ are obtained, especially concerning their derivation algebras $\mathfrak{D}(\mathfrak{A}_t)$. These for $t=2, 3$ are known, $\mathfrak{D}(\mathfrak{C})$ for F not of characteristic 2 having dimension 14 over F, a central simple Lie algebra of type G in case F has characteristic 0 [Jacobson, Duke Math. J. **5**, 775–783 (1939); these Rev. **1**, 100]. The author's main result states that for F not of characteristic 2 or 3 the derivation algebra of any $\mathfrak{A}_t$ $(t>3)$ is isomorphic to that of the Cayley-Dickson subalgebra $\mathfrak{C}=\mathfrak{A}_3$ from which it was constructed; this $\mathfrak{C}$ is invariant under $\mathfrak{D}(\mathfrak{A}_t)$, and any derivation of $\mathfrak{C}$ can be extended in a unique way to a derivation of $\mathfrak{A}_t$. Whereas a Cayley-Dickson algebra $\mathfrak{C}$ has the property that any generalized quaternion subalgebra $\mathfrak{Q}$ of it may be regarded as the $\mathfrak{A}_2$ from which it was constructed, it is shown that the analogous property is not shared by $\mathfrak{A}_t$ generally, at least when F has characteristic 0. *I. M. H. Etherington.*

Kowalsky, Hans-Joachim. Beiträge zur topologischen Algebra. Math. Nachr. 11, 143–185 (1954).

This Habilitationsschrift begins with several sections on a formalism in the subject of filters that can be applied to abbreviate some definitions and theorems on topological spaces, topological groups, etc. The last half of the paper is concerned with the lattice of topologies on a division ring K that are compatible with the ring structure. (The lattice ordering $\leqq$ in this paper means "is finer than".) Every topology except the unit element of the lattice (the trivial topology with no proper open sets) is necessarily Hausdorff.

Any coarsest nontrivial topology is shown to have the property that the completion of K is simple (hence, a field in the commutative case). If such a coarsest topology is locally bounded then it is of type V (hence a valuation topology in the commutative case). In fact, the author proves that for every locally bounded topology there is a type V topology which is coarser. This last results from two facts: locally bounded topologies are given by "Fastordnungen" [Kowalsky and Dürbaum, J. Reine Angew. Math. **191**, 135–152 (1953); these Rev. **15**, 98] and a theory of Fastordnungen can be constructed generalizing Krull's embedding of an order in a valuation ring. The author also constructs a theory of integral closure for Fastordnungen that arrives at the theorem: A Fastordnung is integrally closed if and only if it is the intersection of Vollfastordnungen (the generalization of valuation rings).

Some final theorems concern topologies which are meets of locally bounded topologies. In particular a locally bounded topology τ is the meet of locally bounded topologies τ_α if and only if τ is the meet of finitely many τ_α's. This gives easy examples of non-locally bounded topologies on fields. The paper ends with a statement of some unsolved problems. *D. Zelinsky* (Evanston, Ill.).

Theory of Groups

Gacsályi, S. On algebraically closed abelian groups. Publ. Math. Debrecen **2**, 292–296 (1952).

Let A be an abelian group admitting the ring R with unit element as left operator domain. The author considers infinite systems of linear equations of the form

$$r_1x_{\nu_1}+r_2x_{\nu_2}+\cdots+r_kx_{\nu_k}=c,$$

where $r_i \in R$, $c \in A$, and x_ν, ν in some index set, are indeterminates. Such a system has a solution in A provided that every finite subsystem has, under conditions which include the two cases (i) A is completely divisible and R is the ring of integers, and (ii) A is the additive group of a skew field and R is its ring of left multiplications. An example shows that no such theorem is true in general.
Graham Higman (Manchester).

Szele, T. On arbitrary systems of linear equations. Publ. Math. Debrecen **2**, 297–299 (1952).

An independent treatment of case (ii) in the preceding review. *Graham Higman* (Manchester).

Kertész, A. On subgroups and homomorphic images. Publ. Math. Debrecen **3** (1953), 174–179 (1954).

An abelian group is free abelian if and only if every abelian group of which it is a homomorphic image has a subgroup isomorphic to it. Dually, an abelian group is complete if and only if it is a homomorphic image of every abelian group which has it as a subgroup. An abelian group is a direct sum of a free abelian group and a complete group if and only if every abelian group which has it as an endomorphic image has a direct summand isomorphic to it. The first proposition remains true if the word abelian is omitted throughout; but a group having properties similar to those enunciated in the second and third sentences, but relative to the set of all groups, must be the trivial group. The author poses the question: Is there any non-cyclic finite group such that every finite group of which it is a homomorphic image has a subgroup isomorphic to it?
Graham Higman (Manchester).

Fuchs, L., Kertész, A., and Szele, T. Abelian groups in which every serving subgroup is a direct summand. Publ. Math. Debrecen **3** (1953), 95–105 (1954).

For an abelian group G, the authors prove that every serving subgroup is a direct summand if and only if $G=A\oplus B$, where A is the direct sum of groups of type p^∞ and/or of copies of the additive group R of rationals; and B is (i) the direct sum of cyclic p-groups where the orders of the summands are bounded for each p, or B is (ii) the direct sum of a finite number of copies of some proper subgroup of R. If A is not periodic, (i) is not possible. An abelian group G has every subgroup serving if and only if G is elementary abelian. *F. Haimo* (St. Louis, Mo.).

Szélpál, I. The abelian groups with torsion-free endomorphism ring. Publ. Math. Debrecen **3** (1953), 106–108 (1954).

The author proves that the additive group of the endomorphism ring of an abelian group G is aperiodic if and only if $G=A\oplus B$ where A is a completely divisible periodic group and B is an aperiodic group with the property $pB=B$ for every prime p which appears as a divisor of the order of some element of A. If G is periodic, the condition simplifies to G being the direct sum of groups of type p^∞.
F. Haimo (St. Louis, Mo.).

Szép, J. Bemerkung zu einem Satz von O. Ore. Publ. Math. Debrecen **3** (1953), 81–82 (1954).

The author proves that a finite group G is not simple if and only if it has a proper subgroup H which commutes with all the conjugates in G of a p-Sylow subgroup of H and which also commutes with all the conjugates of a subgroup L, the order of which is not divisible by p. This improves a result of Ore [Duke Math. J. **5**, 431–460 (1939), Theorem 16, p. 438]. *F. Haimo* (St. Louis, Mo.).

Zacher, Giovanni. Determinazione dei gruppi d'ordine finito relativamente complementati. Rend. Accad. Sci. Fis. Mat. Napoli (4) **19** (1952), 200–206 (1953).

The author characterizes all finite groups whose subgroup lattice is relatively complemented, that is, for which to every pair of subgroups A and B, $A\subset B$, and every subgroup X between A and B there exists another subgroup X' such that $X\cap X'=A$ and $X\cup X'=B$. These groups turn out to be precisely so-called t-groups (groups in which the property of being a normal subgroup is transitive) in which all Sylow-subgroups are elementary abelian. *K. A. Hirsch.*

Griffiths, H. B. A note on commutators in free products. Proc. Cambridge Philos. Soc. **50**, 178–188 (1954).

Following Graham Higman [J. London. Math. Soc. **27**, 73–81 (1952); these Rev. **13**, 623] the unrestricted free product of a sequence $\{G_n\}$ of groups is defined. This is the inverse limit K of the free products K_n of $G_1, \cdots, G_n$ with the homomorphisms $K_{n+1}\rightarrow K_n$ determined by $G_{n+1}\rightarrow 1$. It is shown that in the natural topology for K the derived group $[K, K]$ is not closed. Also, the subgroup $P(K)$ obtained as the limit of words

$$g_n=a_1a_2\cdots a_{m-1}a_m,\quad g_{n+1}=a_1{}^1a_1a_2{}^1a_2\cdots a_ma^1{}_{m+1}$$

with $a_1{}^1 \in G_{n+1}$ is not even normal in K. These results rest on showing that in an ordinary free product the product of more than $12n-2$ elements from distinct G's is not a product of n commutators. *Marshall Hall, Jr.*

Hall, Marshall, Jr. On a theorem of Jordan. Pacific J. Math. 4, 219–226 (1954).

A group quadruply transitive on a set of letters, finite or infinite, in which a subgroup H fixing four letters is of finite odd order, must be one of the following: the symmetric group on four or five letters, the alternating group on six or seven letters, or the Mathieu group on eleven letters.

Graham Higman (Manchester).

Baer, Reinhold. Das Hyperzentrum einer Gruppe. IV. Arch. Math. 5, 56–59 (1954).

[For parts I–III see these Rev. 15, 395, 396, 598.] If $\mathfrak{G}$ is a normal subgroup of a group $\mathfrak{E}$, the restrictions to $\mathfrak{G}$ of the inner automorphisms of $\mathfrak{E}$ form a group G of automorphisms of $\mathfrak{G}$ containing the inner automorphisms. The question whether $\mathfrak{G}$ is upper hypercentral, or hypercentral, in $\mathfrak{E}$ [cf. Baer, loc. cit., part II] depends only on G. In fact, $\mathfrak{G}$ is upper hypercentral if and only if for each normal subgroup $\mathfrak{N}(\neq\mathfrak{G})$ of $\mathfrak{G}$ admitting G, the automorphism group of $\mathfrak{G}/\mathfrak{N}$ induced by G has a fixed element other than the identity; and $\mathfrak{G}$ is hypercentral if and only if for each subgroup U of G, and each pair $\mathfrak{V}$, $\mathfrak{W}$ of distinct subgroups of $\mathfrak{G}$ which admit U and are such that $\mathfrak{W}$ is normal and of finite index in $\mathfrak{V}$, the automorphism group of $\mathfrak{V}/\mathfrak{W}$ induced by U has a fixed element other than the identity.

Graham Higman (Manchester).

Čarin, V. S. On groups of automorphisms of certain classes of solvable groups. Ukrain. Mat. Žurnal 5, 363–369 (1953). (Russian)

Let the subgroup Γ of the group of all automorphisms of a group $\mathfrak{G}$ be a complete group. If $\mathfrak{G}$ is solvable and has finite rank, then Γ is solvable. If $\mathfrak{G}$ is nilpotent and has finite rank, then Γ is nilpotent. If $\mathfrak{G}$ is a solvable group of type A_4, then Γ is nilpotent, is torsion-free, and has finite rank. If $\mathfrak{G}$ is a solvable group of type A_3, then Γ is the extension of a complete abelian group by a complete nilpotent torsion-free group of finite rank. The nomenclature for types A_3 and A_4 has been described by Mal'cev [Doklady Akad. Nauk SSSR (N.S.) 67, 23–25 (1949); these Rev. 11, 78].

R. A. Good (College Park, Md.

Lyndon, R. C. On the Fouxe-Rabinovitch series for free groups. Portugaliae Math. 12, 115–118 (1953).

With any word $w=x_{\omega_1}^{\epsilon_1}\cdots x_{\omega_m}^{\epsilon_m}$ (each $\epsilon_i=\pm1$) representing an element of the free group F on the generators x_i the author associates the formal power series

$$\pi(w)=\sum\pi_\sigma(w)x_{\sigma_1}\cdots x_{\sigma_n}$$

in the non-commuting variables x_i, where the coefficients $\pi_\sigma(w)$ are the "determinators" introduced by Fouxe-Rabinowitch [Mat. Sbornik N.S. 7(49), 197–208 (1940); these Rev. 2, 1]: $\pi_\sigma(w)=\sum\epsilon(\theta)\varphi(\theta)$ summed over all sequences $\theta=\theta_1, \cdots, \theta_n$ $(1\leqq\theta_i\leqq m)$ such that $\omega_{\theta_i}=\sigma$ for each i, $\epsilon(\theta)=\epsilon_{\theta_1}\cdots\epsilon_{\theta_n}$ and $\varphi(\theta)=\prod_{i=1}^{r-1}\operatorname{sgn}(\theta_{i+1}-\theta_i)$. Here $\pi(w)$ depends only on the element of F represented by w. The author shows that, for $u,v\in F$,

$$L+\pi(uv)=[1+\pi(u)][1+\pi(v)\pi(u)]^{-1}[1+\pi(v)]$$

and indicates the connection of the series $\pi(w)$ with the analogous series $D(w)$ of Magnus, which has the composition law $1+D(uv)=[1+D(u)][1+D(v)]$.

P. Hall.

NUMBER THEORY

Arcidiacono, Giuseppe. Sulla estensione delle operazioni aritmetiche. Collectanea Math. 6, 91–105 (1953).

The first part of this paper discusses the general arithmetic operation of the nth order. For $n=1$, 2 and 3 these are addition and multiplication and exponentiation, respectively. There is a definition of the inverse operations corresponding to radicals and logarithms. The second part deals in detail with the case $n=4$. Here the direct operation is

$$\overline{a}|^b=a^{a^{b-1}}.$$

The inverse of this function for $b=2$ is called the hyper square root. This is discussed and illustrated by a 6-decimal table of the real roots y of $y^y=x$ for $x=1(1)40(50)60$, $99(1)105$ taken from a table of Eisenstein [J. Reine Angew. Math. 28, 49–52 (1844)]. The concepts and notations are applied to the solution of a number of equations of various types.

D. H. Lehmer (Berkeley, Calif.).

Bieberbach, Ludwig. Über Stifelsche magische Quadrate. I. Arch. Math. 5, 4–11 (1954).

Considering a magic square as necessarily consisting of consecutive integers, a Stifel magic square has the property that if m rows are removed from the top, bottom, right and left edges the resulting square, if of order >2, is magic. In this paper various results are obtained concerning the possible arrangements of the numbers in the outer rows of a Stifel square.

R. J. Walker (Ithaca, N.Y.).

Storchi, Edoardo. Risoluzione generale in interi dell'equazione:

$$\operatorname{arctg}\frac{m}{n}=\operatorname{arctg}\frac{1}{x}+\operatorname{arctg}\frac{1}{y}.$$

Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 191–206 (1952).

Integral solutions (x, y) are sought for the equation $\arctan(1/x)+\arctan(1/y)=\arctan(m/n)$, where m and n are given relatively prime positive integers. This equation can be rewritten in the form $(mx-n)(my-n)=m^2+n^2$, whence it follows that for each factorization $q\bar{q}=m^2+n^2$, $0<q\leqq\bar{q}$, we have a solution if either $n+q$ or $n-q$ is divisible by m. When $m=1$, 2, 3, 4, or 6, such solutions exist for all n relatively prime to m, but for other values of m there are values of n for which no integral solutions exist, in particular if $n<m/3$. Similar arctangent relations have been discussed by Lehmer [Amer. Math. Monthly 45, 657–664 (1938)] and Todd [ibid. 56, 517–528 (1949); these Rev. 11, 159].

J. S. Frame (East Lansing, Mich.).

Hemer, Ove. Notes on the Diophantine equation $y^2-k=x^3$. Ark. Mat. 3, 67–77 (1954).

This paper gives some additions and corrections to the author's dissertation [Univ. of Uppsala, 1952; these Rev. 14, 354]. In addition to refinements of the theory, there are corrections in application of the theory in a few instances, and the removal of one or two numerical errors. The Diophantine equation $y^2-k=x^3$ is now solved completely in all cases $0<k\leqq100$, and in all but 22 of the cases $0<-k\leqq100$.

Corrected tables of solutions and of the fundamental units of the cubic fields are given. *I. Niven* (Eugene, Ore.).

Brauer, Alfred, and Seelbinder, B. M. On a problem of partitions. II. Amer. J. Math. 76, 343–346 (1954).

The linear Diophantine equation in k unknowns

$$a_1x_1+a_2x_2+\cdots+a_kx_k=n,$$

where the letters represent positive integers and the a's have no common factor, has a solution for n sufficiently large. This paper is a continuation of a previous one [Brauer, same J. 64, 299–312 (1942); these Rev. 3, 270; 5, 328] in which bounds $F(a_1, \cdots, a_k)$ such that $n>F$ implies that the equation is solvable are discussed. One bound proposed is

$$T(a_1, \cdots, a_k)=\sum_{\nu=2}^{k} a_\nu d_{\nu-1}/d_\nu,$$

where d_ν is the greatest common divisor of the first ν a's. It was shown previously that in case the a's are such that

$$(1)\qquad a_\nu/d_\nu=\left(\sum_{\lambda=1}^{\nu-1} a_\lambda y_\lambda\right)\Big/ d_{\nu-1},$$

where the y's are non-negative integers, then T is the best bound possible. In the present paper it is shown conversely that (1) is a necessary condition for T to be best possible. In case (1) does not hold, a better bound is given by $T-\min(a_1, a_2, \cdots, a_k)$. *D. H. Lehmer.*

Ankeny, N. C. Quadratic residues. Duke Math. J. 21, 107–112 (1954).

Denote by $n(p)$ the least quadratic nonresidue (mod p), where p denotes a prime $\equiv 3$ (mod 4). The author proves that for any ϵ and $p>p_0(\epsilon)$

$$(1)\qquad n(p)<p^\epsilon.$$

(1) probably holds for all primes. Denote by $h(p)$ the number of classes of ideals in the field $R(\sqrt{-p})$. Let us assume that all integers $1\leqq u<p^\epsilon$ are quadratic residues of p. Then using a result of Linnik and Rényi [Izvestiya Akad. Nauk SSSR. Ser. Mat. 11, 539–546 (1947); these Rev. 9, 333], the author shows that

$$(2)\qquad h(p)=o(p[\log p]^{1/2}).$$

Next he shows that

$$(3)\qquad h(p)\geqq \sum_{1\leqq n\leqq \frac{1}{2}p^{1/2}} \bar{d}(n, p^\epsilon)$$

where $\bar{d}(n, p)=0$ if n has a prime factor greater than p^ϵ and if all prime factors of n are $\leqq p^\epsilon \bar{d}(n, p^\epsilon)=2^{\nu(n)}$ where $\nu(n)$ denotes the number of prime factors of n. Using some results of de Bruijn [Nederl. Akad. Wetensch. Proc. Ser. A. 54, 50–60 (1951); these Rev. 13, 724], the author shows that (3) contradicts (2); thus (1) is proved. *P. Erdös.*

Carlitz, Leonard. Weighted quadratic partitions (mod p^r). Math. Z. 59, 40–46 (1953).

The results of this paper are generalizations to $r>1$ of some of the results for $n=1$ obtained by the author in a previous paper [Canadian J. Math. 5, 317–323 (1953); these Rev. 15, 508]. They are explicit evaluations of

$$S=\sum_{x_1,\cdots,x_t} e_r(2\lambda_1x_1+\cdots+2\lambda_tx_t),$$

summed over all integers $x_1, \cdots, x_t$ (mod p^r) such that

$$a_1x_1^2+\cdots+a_tx_t^2\equiv c \pmod{p^r},$$

where $e_r(x)=\exp(2\pi ix/p^r)$, and the λ's, a's and c are fixed integers, with all the a's prime to p and not all the λ's divisible by p. *R. Hull* (Lafayette, Ind.).

Carlitz, L. Hankel determinants and Bernoulli numbers. Tôhoku Math. J. (2) 5, 272–276 (1954).

The Hankel determinant $\Delta=|a_{i+j}|$ of order r satisfies the relation $\Delta=|\Delta^{i+j}a_0|$, where $\Delta^r a_i$ is the rth difference of a_i in an arbitrary sequence $\{a_i\}$. If the a_i are rational numbers that are integral modulo some fixed integer M and such that $\Delta^r a_0\equiv 0$ (mod M^r), then $\Delta\equiv 0$ (mod $M^{r(r+1)}$). There are constructed in this paper several examples of Hankel determinants of Bernoulli, Euler and related numbers that satisfy corresponding congruence properties.

A. L. Whiteman (Los Angeles, Calif.).

Carlitz, L. The number of solutions of some special equations in a finite field. Pacific J. Math. 4, 207–217 (1954).

Let $N_f(\alpha)$ denote the number of solutions of the equation $f=\alpha$, where f is a polynomial in r variables taken over a field k with q elements. In this paper there are furnished several instances of polynomials f having the property that $N_f(\alpha)=N_f(1)$ for all $\alpha\neq 0$. A typical theorem states in part that if q is odd and if P denotes the general Pfaffian in $r(2r-1)$ indeterminates, then the number of solutions of the equation $P=\alpha$ is given by

$$N_P(\alpha)=q^{r(r-1)}\prod_{i=1}^{r}(q^{2i-1}-1)\qquad(\alpha\neq 0).$$

Another theorem gives the number of solutions of

$$F_1(x^{(1)})+\cdots+F_s(x^{(s)})=\alpha,$$

where each F is homogeneous and irreducible and factors completely into linear factors in some extension field of k.

A. L. Whiteman (Los Angeles, Calif.).

Carlitz, L. Certain special equations in a finite field. Monatsh. Math. 58, 5–12 (1954).

This paper concerns equations such as

$$ax^2+by^2+cz^2=2dxyz+e$$

taken over a field k with q elements, q being odd. It is proved that the number of solutions (x, y, z) of the above equation subject to the condition $abcd\neq 0$ is determined by

$$q^2+1+q\{\psi(a)+\psi(b)+\psi(c)+\psi(e)\}\psi(d^2e-abc),$$

where $\psi(a)=+1, -1, 0$ according as a is a square, a non-square or zero in k. The expression for the number of solutions of the corresponding equation in four variables depends upon the values of certain Jacobsthal sums.

A. L. Whiteman (Los Angeles, Calif.).

Carlitz, L. Representations by skew forms in a finite field. Arch. Math. 5, 19–31 (1954).

Let $q=p^n$, $p>2$, and let A, B be skew-symmetric matrices of order m and t, respectively, with elements in GF(q). The author investigates the problem of finding the number $Z_t(A, B)$ of $m\times t$ matrices X with elements in GF(q) such that $X'AX=B$, where X' denotes the transpose of X. The general case is reduced to the simpler cases in which (1) A is non-singular of order $2m$, B non-singular of order $2r$, and (2) $B=0$. In case (1) the author proves the formula

$$Z_{2r}(A, B)=q^{m-r^2}\prod_{i=0}^{r-1}(q^{m-2i}-1).$$

In case (2) he obtains a different type of formula which

exhibits $Z_t(A, 0)$ as the sum of a terminating q-hypergeometric series of type ${}_2\Phi_0$. The author also obtains incidentally a formula for the number of skew-symmetric matrices of order t and rank $2r$. Finally he applies his results to solve the problem of determining the number of solutions $X=X(m, t)$ of the equation $D=\rho$, where D is a determinant of order 2 having the elements A, X in its first row and the elements X', 0 in its second.

A. L. Whiteman (Los Angeles, Calif.).

Carlitz, L. A problem involving quadratic forms in a finite field. Math. Nachr. 11, 135–142 (1954).

Let $q=p^n$, $p>2$ and let GF(q) denote the finite field of order q. Let $A=(\alpha_{ij})$ denote a symmetric matrix of order m with elements in GF(q) such that $\delta=|\alpha_{ij}|\neq 0$. Let r and t be arbitrary integers $\geqq 1$ and consider the equation

$$(*)\qquad \begin{vmatrix} A_1 & X_1 \\ X_1' & 0 \end{vmatrix} + \cdots + \begin{vmatrix} A_r & X_r \\ X_r' & 0 \end{vmatrix} = \beta,$$

where β is an assigned number of GF(q), the A_i are assigned non-singular matrices of order m_i, and the X_i are $m_i\times t_i$ matrices. In the case $r=1$ the author reduces the problem of deriving the number $N_{m,t}(A,\beta)$ of solutions X of (*) to a special case of one discussed in one of his previous papers [Duke Math. J. 21, 123–137 (1954); these Rev. 15, 604]. The results appear as explicit product formulas depending on both the parity of m and t. When r is an arbitrary integer the determination of the number of solutions $X_1, \cdots, X_r$ of (*) is made to depend upon the evaluation of a generalized Gauss sum. *A. L. Whiteman.*

Carlitz, L. Note on irregular primes. Proc. Amer. Math. Soc. 5, 329–331 (1954).

A prime p is irregular if it divides the numerator of at least one of the numbers $B_2, B_4, \cdots, B_{p-3}$, where B_m denotes a Bernoulli number in the even-suffix notation. Jensen has proved that there exist infinitely many irregular primes of the form $4n+3$ [for the proof see H. S. Vandiver, Bull. Nat. Res. Council no. 62, 28–111 (1928), p. 82]. In this note the author gives a simple proof of the weaker result that the number of irregular primes is infinite. He also proves a like result corresponding to the prime divisors of Euler numbers. *A. L. Whiteman.*

Weil, André. Footnote to a recent paper. Amer. J. Math. 76, 347–350 (1954).

In a recent paper Carlitz [same J. 76, 137–154 (1954); these Rev. 15, 404] has given formulas for the number of solutions of special pairs of equations $F=a$, $G=b$, where F, G are quadratic forms over a field k with q elements, q being odd. The present author derives more complete results briefly by the same method. He also points out that his results provide additional evidence concerning the conjectures on the zeta-functions of nonsingular varieties formulated in one of his earlier articles [Bull. Amer. Math. Soc. 55, 497–508 (1949); these Rev. 10, 592].

A. L. Whiteman (Los Angeles, Calif.).

Samet, P. A. Algebraic integers with two conjugates outside the unit circle. II. Proc. Cambridge Philos. Soc. 50, 346 (1954).

[For part I see same Proc. 49, 421–436 (1953); these Rev. 15, 14.] Let S_2' be the set of real conjugate integers ρ, σ, such that their algebraic conjugates except ρ and σ satisfy $|z|<1$. The author proves that all quadratic integers of S_2' are limit points of S_2'. *H. Bergström* (Göteborg).

Zulauf, Achim. Über die Darstellung natürlicher Zahlen als Summen von Primzahlen aus gegebenen Restklassen und Quadraten mit gegebenen Koeffizienten. I. Resultate für genügend grosse Zahlen. J. Reine Angew. Math. 192, 210–229 (1953).

The author obtains a generalization of a theorem of Stanley [Proc. London Math. Soc. (2) 29, 122–144 (1929)] and Halberstam [ibid. 53, 363–380 (1951); these Rev. 13, 112] by a method of Linnik [Mat. Sbornik N.S. 19 (61) 3–8 (1946); these Rev. 8, 317]. The investigation of the singular series involved is to follow in another paper.

T. Estermann (London).

Lehmer, D. H., Lehmer, Emma, and Vandiver, H. S. An application of high-speed computing to Fermat's last theorem. Proc. Nat. Acad. Sci. U. S. A. 40, 25–33 (1954).

In connection with the proof of Fermat's Last Theorem, it is necessary to investigate the regularity or irregularity of the prime p which appears as the exponent in $x^p+y^p=z^p$. A prime p is regular if it does not divide any of the first $(p-3)/2$ Bernoulli numbers. The truth of Fermat's conjecture for the case of regular primes was settled by Kummer [J. Reine Angew. Math. 40, 93–116, 117–129, 130–138 (1850)]. Kummer also devised criteria which apply to some irregular primes. In the years 1928–1936, H. S. Vandiver and a number of collaborators developed further criteria for irregular primes. Since the arithmetic labor of examining these criteria was enormous, the calculations were carried out only for the 36 irregular primes less than 619.

The purpose of the present note is to indicate how the original criteria of Vandiver can be adapted to the SWAC (the high-speed computing machine at the Institute for Numerical Analysis at UCLA). The coding was performed by D. H. and Emma Lehmer. The result of the calculation is that Fermat's Last Theorem is true for all primes less than 2000. However, as the authors point out, calculations of this type are of great use in the theory of cyclotomic fields, regardless of their applicability to Fermat's problem.

R. Bellman (Santa Monica, Calif.).

Vandiver, H. S. New types of trinomial congruence criteria applying to Fermat's last theorem. Proc. Nat. Acad. Sci. U. S. A. 40, 248–252 (1954).

The criteria discussed are concerned with the second case of Fermat's Last Theorem. An attempt is made to obtain conditions which involve the exponent l in

$$(1)\qquad x^l+y^l+z^l=0$$

rather than x, y or z. The main result is as follows. Let (1) be solvable in non-zero integers one of which is divisible by the prime l. Let p be a prime of the form $1+cl$ not dividing xyz. Let q be a primitive root of p and let the exponent of the prime ideal in $K(\exp 2\pi i/l)$ dividing the ideal (p) not be a multiple of l. Then there are integers i and j such that the number (ij) of solutions α, β of the congruence

$$1+g^{i+c\alpha}\equiv g^{j+l\beta} \pmod P$$

is equal to l. Here g is a primitive root of p. Since $\sum_{j=0}^{l-1}(ij)\leqq l$, the criterion $(i,j)=l$ is indeed a strong one. Unfortunately it is not strong enough to be impossible. It holds for $l=3$, $p=109$ and $l=5$, $p=61051$. No further cases with $l>5$ are known. *D. H. Lehmer* (Berkeley, Calif.).

★**Pellegrino, Franco. Sviluppi moderni del calcolo numerico integrale di Michele Cipolla.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 161–168. Casa Editrice Perrella, Roma, 1953.

This paper is devoted to a subjective exposition of the work of M. Cipolla [Revue Math. (Revista Mat.) 9 (1909); Atti Accad. Gioenia Catania (5) 8, no. 11 (1915)] on arithmetical functions and their generating functions. A number of theorems about the Cauchy product

$$h(n)=(f\times g)(n)=\sum f(d)g(n/d),$$

where the sum extends over all the divisors of n, and its connection with Dirichlet series are attributed to Cipolla and set forth without proof. The main result states that if $h(n)$ and $g(n)$ are given functions with $g(1)\neq 0$, then there exists a uniquely determined function f such that $f\times g=h$. In the latter part of the paper the theorems of Cipolla are extended and reformulated in the framework of the terminology of modern algebra and topology. The results are too complicated to be summarized conveniently. Further details and proofs are to appear in a later paper.

A. L. Whiteman (Los Angeles, Calif.).

Gupta, H., Cheema, M. S., and Gupta, O. P. On Möbius means. Res. Bull. Panjab Univ. no. 42, 16 pp. (1 plate) (1954).

The authors give tables which serve to test the conjectures of Brun and Siegel about the function $\mu_2(n)=\sum_{k=1}^{n}\mu_1(n)$, where $\mu_1(n)=\sum_{k=1}^{n}\mu(n)$, for the Möbius function μ. The conjecture is that on the average

$$n^{-1}\mu_2(n-1)=-2+12n^{-1}-\beta n^{-2},$$

where $\beta=18$ according to Brun and $\beta=2\pi^2/\zeta(3)=16.421$ according to Siegel. [See Brun, C. R. Dixième Congrès Math. Skandinaves, 1946, Gjellerup, Copenhagen, 1947, pp. 40–53; these Rev. 8, 446; Gupta, J. Indian Math. Soc. (N.S.) 13, 85–90 (1949); these Rev. 11, 234.] The authors tabulate

$$F(n)=T_0(n)=n\mu_2(n-1)+2n^2-12n$$

and its successive means

$$T_k=n^{-1}\sum_{\nu=1}^{n}T_{k-1}(\nu)\quad(k=1,2,3,4,5)$$

for $n=1(1)750$. Over the last half of the table $T_5(n)$ descends gradually from -14 to -15.

D. H. Lehmer.

Kotelyanskiĭ, D. M. On N. P. Romanov's method of obtaining identities for arithmetic functions. Ukrain. Mat. Žurnal 5, 453–458 (1953). (Russian)

The author obtains a variety of identities by specialising in an identity of Romanov [Izvestiya Akad. Nauk. SSSR. Ser. Mat. 10, 3–34 (1946); these Rev. 8, 9]. Thus he shows that, under certain convergence conditions,

$$\sum_{n=1}^{\infty}\frac{F(n)G(n)}{n^s}=\zeta(s)\sum_{n=1}^{\infty}\frac{\varphi_s(n)}{n^{2s}}\sum_{k=1}^{\infty}\frac{f(nk)}{k^s}\sum_{l=1}^{\infty}\frac{g(nl)}{l^s},$$

where $F(n)$ and $G(n)$ are multiplicative functions, and

$$F(n)=\sum_{d|n}f(d),\quad G(n)=\sum_{d|n}g(d).$$

Also

$$\varphi_s(n)=n^s\prod_{p|n}\left(1-\frac{1}{p^s}\right).$$

[Note: in the paper, formula (13), n^s is printed instead of n^{2s}, on the right.] By choosing $f(n)$ and $g(n)$ in various ways a variety of identities, some of which are known, can be obtained.

R. A. Rankin (Birmingham).

Deuring, Max. Die Zetafunktion einer algebraischen Kurve vom Geschlechte Eins. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. 1953, 85–94 (1953).

Let k be an algebraic number field and let C be a plane algebraic curve over k defined by the (absolutely irreducible) equation $f(x,y)=0$ with coefficients in k; let C be of genus g. Let p denote a prime divisor of k. Then $f(x,y)$ is absolutely irreducible and defines an algebraic curve C/p of genus g for almost all p. Define the zeta-function

$$\zeta(s,C,k,p)=\sum_a\frac{1}{(Na)^s}\quad(\Re s>1),$$

where a runs through the integral divisors of $(k/p)(x,y)$. Then it is known that

$$\zeta(s,C,k,p)=\frac{L(s,C,k,p)}{(1-Np^{-s})(1-Np^{1-s})},$$

where $L(s,C,k,p)$ is a polynomial in Np^{-s} of degree $2g$; moreover,

$$L(s,C,k,p)=\prod_{r=1}^{2g}(1-\pi_rNp^{-s})\quad(|\pi_r|=Np^{1/2}).$$

Now define

$$\zeta(s,C,k)=\zeta(s,k)\zeta(s-1,k)\prod_p L(s,C,k,p),$$

where $\zeta(s,k)$ is the Dedekind zeta-function. Then $\zeta(s,C,k)$ is regular for $\Re s>3/2$ except for a pole at $s=2$. For $g=0$ it is easy to extend the function to the entire s-plane and to show that it satisfies a functional equation of the usual kind. It is therefore natural to ask whether the same is true for $g\geqq 1$. A. Weil [Trans. Amer. Math. Soc. 73, 487–495 (1952); these Rev. 14, 452] showed that for the equation $ax^m+by^n+c=0$, the product $\prod L(s,C,k,p)^{-1}$ is the product of $2g$ Hecke L-series of argument $s-\frac{1}{2}$. He conjectured that this result holds when $g=1$ in the case of complex multiplication. The truth of this conjecture is proved in the present paper.

L. Carlitz (Durham, N. C.).

Hasse, Helmut. Über das Zerlegungsgesetz für einen Funktionalprimdivisor in einem zyklischen Körper von durch ihn teilbarem Primzahlpotenzgrad. Arch. Math. 5, 216–225 (1954).

In a recent paper [Trans. Amer. Math. Soc. 73, 487–495 (1952); these Rev. 14, 452] Weil constructed a zeta-function for the function-field defined by $ax^m+by^n=c$ and showed that it can be expressed as the product of $2g$ Hecke L-functions, where g is the genus of the function-field. This result suggests the following problem. Let K denote an algebraic number field of finite degree over the rational field R and let $\mathfrak{p}$ be a prime divisor of K; let $\mathfrak{p}$ divide the rational prime p. Assume that $K\supset Z_n$, the field of the p^nth roots of unity. If t is an indeterminate, then the mapping $t\to t^p$ defines a meromorphism P of $R(t)$ onto $R^P=R(t^p)$. Let $u\in R(t)$, u integral (mod p), and define

$$K_n=K(t,(u^{P^n})^{p^{-n}}).$$

It is also assumed that if $u^p=u^P+u'p$, then u' is not a pth power residue (mod p). For example this condition is satisfied if $u=1+ct^m$, $p\nmid m$ and c a rational number prime to p. The author now raises the question of describing the prime factor $\mathfrak{p}_n$ of $\mathfrak{p}$ in the field K_n. We quote only the first theorem of the paper. Let ζ_ν denote a primitive p^νth root

of unity, $\zeta_\nu^p=\zeta_{\nu-1}$ and put $\pi_\nu=1-\zeta_\nu$; let π_n be divisible by exactly $\mathfrak{p}^e$, thus defining e. Then we have the following results. If $p\nmid e$ then $\mathfrak{p}=\mathfrak{p}_1^p$. If $p|e$ then $p=\mathfrak{p}$.

L. Carlitz (Durham, N. C.).

Ollerenshaw, Kathleen. Irreducible convex bodies. Quart. J. Math., Oxford Ser. (2) 4, 293–302 (1953).

A star body S is defined to be reducible if a star body T exists which is properly contained within S and for which the critical determinants $\Delta(S)$ and $\Delta(T)$ are the same. Let $f(x_1, x_2)\leqq 1$, define an irreducible convex plane domain. The author shows that both the n-dimensional sphere, $n\leqq 5$, and the generalized cylinder $f(x_1, x_2)\leqq 1$, $|x_3|\leqq 1, \cdots, |x_n|\leqq 1$ are irreducible star bodies. Rogers [Nederl. Akad. Wetensch., Proc. 50, 868–872 (1947); these Rev. 9, 228] has proved that a star body S is irreducible if each of its boundary points P satisfies the following condition. For every ϵ, $\epsilon>0$, a lattice Λ exists with $\Delta(\Lambda)=\Delta(S)$ whose only points interior to S are 0, $\pm P^*$ where $|P-P^*|<\epsilon$. The results in the paper reviewed are obtained by constructing lattices Λ for boundary points of the given star bodies.

D. Derry.

Kneser, Martin. Zur Theorie der Kristallgitter. Math. Ann. 127, 105–106 (1954).

The author gives a shorter and more constructive proof of Theorems 1 and 2 of a paper by Eichler [Math. Ann. 125, 51–55 (1952); these Rev. 14, 851].

R. Hull.

Bambah, R. P. On lattice coverings by spheres. Proc. Nat. Inst. Sci. India 20, 25–52 (1954).

In this paper the author obtains, for the first time, the most economical lattice coverings of three-dimensional Euclidean space by equal spheres. Let Λ be a lattice, of determinant $d(\Lambda)$, and suppose that equal spheres K of volume $V(K)$ are centered at each point of Λ, and that each point of the space belongs to at least one such sphere. Define $\theta(K)$ to be the lower bound of $V(K)/d(\Lambda)$, taken over all Λ. Then $\theta(K)$ is the density of the most economical lattice covering by spheres K and is shown by the author to be $5^{3/2}\pi/24$. Further, a lattice Λ provides the most economical lattice covering by K if and only if, by a suitable choice of orthogonal axes, Λ is generated by the points

$$\frac{2r}{\sqrt5}(-1, 1, 1),\quad \frac{2r}{\sqrt5}(1, -1, 1),\quad \frac{2r}{\sqrt5}(1, 1, -1),$$

and K has the equation $x^2+y^2+z^2=r^2$. This is the body-centred cubic lattice.

This result is equivalent to the following theorem: Let $f(x, y, z)=ax^2+by^2+cz^2+2ryz+2szx+2txy$ be a positive definite quadratic form with real coefficients and determinant $D>0$. Then there exist real numbers x_0, y_0, z_0 such that for all integers we have

$$f(x+x_0, y+y_0, z+z_0)\geqq\left(\frac{125}{1024}D\right)^{\frac13},$$

the sign of equality being necessary if and only if

$$f\sim\rho(3x^2+3y^2+3z^2-2yz-2zx-2xy), \tag{1}$$

and ρ is any positive number.

If D_f is the determinant of a positive definite form f, if $m_f(\xi, \eta, \zeta)$ is the minimum of $f(\xi-x, \eta-y, \zeta-z)$ for all integers x, y, z, and if $M_f=m_fD_f^{-1/3}$, then this theorem is equivalent to the statement $\inf M_f=m=(125/1024)^{1/3}$. To prove this f is taken in Seeber's reduced form, i.e. $0<a\leqq b\leqq c$, $2|s|\leqq a$, $2|t|\leqq a$, $2|r|\leqq b$, $a+b+2r+2s+2t\geqq 0$ and r, s, t are (i) all non-negative or (ii) all negative. Forms of type (i) are first considered and are shown to have an M_f with lower bound greater than m. This is done by considering the various possible relative orders of r, s and t. Forms of type (ii) are then considered. The associated polyhedron Π_f is the region consisting of points X for which $f(X)\leqq f(X-A)$ for each point A with integral coordinates, other than the origin. Π_f is bounded by at most seven pairs of opposite planes and M_f clearly is attained at a vertex, of which there are three types giving values f_1, f_2 and f_3 to f. By considering in detail all the possible relative orderings of f_1, f_2 and f_3, it is shown that a form f for which M_f attains its infimum must have $f_1=f_2=f_3$, and from this it is deduced that f is in fact the form (1).

R. A. Rankin (Birmingham).

Bambah, R. P. Lattice coverings with four-dimensional spheres. Proc. Cambridge Philos. Soc. 50, 203–208 (1954).

Let θ_n be the density of the thinnest lattice covering by spheres in n-dimensional Euclidean space R_n. The author proves that

$$1.5194<\frac{4}{15\sqrt3}\pi^2\leqq\theta_4\leqq\frac{2}{5\sqrt5}\pi^2<1.7656.$$

This is equivalent to the following theorem which he proves: Let

$$f(x)=f(x_1, x_2, x_3, x_4)=\sum_{i,j=1}^{4} a_{ij}x_ix_j\quad(a_{ij}=a_{ji})$$

be a positive definite quadratic form with real coefficients and determinant $D=|a_{ij}|$. Then there exist real numbers u_1, u_2, u_3, u_4 such that for all integers x_1, x_2, x_3, x_4 we have

$$f(x+u)=f(x_1+u_1, x_2+u_2, x_3+u_3, x_4+u_4)\geqq\left(\frac{64^{\frac13}}{675}D\right)^{1/4}.$$

On the other hand, there exists a positive definite quadratic form $f(x)$ such that for all real numbers u_1, u_2, u_3, u_4 we can find integers x_1, x_2, x_3, x_4 with

$$f(x+u)=f(x_1+u_1, x_2+u_2, x_3+u_3, x_4+u_4)\leqq\left(\frac{16}{125}D\right)^{1/4},$$

the equality sign in the last relation being necessary. The proof of the first part depends upon a lemma quoted from the paper reviewed above. It is shown that the last part of the theorem is satisfied by the form

$$f(x)=4\sum_{i=1}^{4}x_i^2-2\sum_{1\leqq i<j\leqq 4}x_ix_j,$$

and it is conjectured that, in fact, $\theta_4=2\pi^2/5\sqrt5$.

R. A. Rankin (Birmingham).

Sawyer, D. B. The lattice determinants of asymmetrical convex regions. J. London Math. Soc. 29, 251–254 (1954).

Let K be an open convex region in Euclidean n-space which contains the origin O. If $P'OP$ be a straight line segment joining the boundary points P', P of K, the coefficient of asymmetry λ of K is defined to be the $\overline{\text{bound}}\ PO/OP'$. The author shows that a region N exists in K, symmetric with respect to the origin, for which

$$V(N)\geqq\left(\frac{2\lambda}{\lambda+1}\right)^n\{\lambda^n-(\lambda-1)^n\}^{-1}V(K),$$

where $V(K)$, $V(N)$ represent the n-dimensional volumes of K and N respectively. This leads to an upper bound of $\overline{\text{bound}}\ V(K)/\Delta(K)$ for a given λ, $\Delta(K)$ being the critical

determinant of the admissible lattices of K. This relationship reduces to the Minkowski inequality $V(K)/\Delta(K) \leq 2^n$ if $\lambda=1$ and K thus becomes symmetric. A lower bound of $V(K)/\Delta(K)$, for a given λ, $\lambda \geq 2$, is obtained by construction of a special asymmetric region K. *D. Derry.*

Oppenheim, A. Criteria for irrationality of certain classes of numbers. Amer. Math. Monthly 61, 235–241 (1954).

The author considers infinite series of the form

$$x=a_0+\frac{a_1}{b_1}+\frac{a_2}{b_1b_2}+\frac{a_3}{b_1b_2b_3}+\cdots,$$

where the a_i and the b_i are integers. The author obtains various criteria for the irrationality of x. Here are some of his theorems. Let $b_i \geq 2$, $0 \leq a_i \leq b_i - 1$; then x is irrational if there exists an irrational α and an infinite sequence i_n so that $a_{i_n}/b_{i_n} \to \alpha$. The number x is irrational if we assume $b_i \geq 1$; further, for every g there is a b_n satisfying $g|b_n$, and finally there is an infinite sequence i_n so that $0<x_{i_n}<1$ (mod 1) where $x_k=a_k/b_k+a_{k+1}/b_k\cdot b_{k+1}+\cdots$. *P. Erdös.*

Pipping, Nils. Über die Elemente der Diagonalkettenbrüche. Acta Acad. Aboensis 19, no. 6, 8 pp. (1954).

Necessary and sufficient conditions for a semi-regular continued fraction to be a diagonal continued fraction are given in terms of the elements of the semi-regular continued fraction. *W. T. Scott* (Evanston, Ill.).

Kuipers, L., and Meulenbeld, B. On a certain classification of the convergents of a continued fraction. II. Nieuw Arch. Wiskunde (3) 2, 32–39 (1954).

The authors continue the investigation begun in part I [same Arch. (3) 1, 199–211 (1953); these Rev. 15, 510], in which irreducible fractions P/Q are separated into 8 classifications determined by the ordered pair $(r(P), r(Q))$ of remainders mod 3 of P, Q. It is shown that for any irrational α the inequality $|\alpha-P/Q|<k/Q^2$ is satisfied by infinitely many P/Q of at least four classes if $k \geq 1$, and of all eight classes if $k>3/\sqrt{5}$. There exist irrationals α for which the inequality is satisfied by infinitely many fractions of only four classes when $k=1$; moreover, there exist irrationals α for which the inequality does not hold for infinitely many fractions of all of the eight classes when $k<3/\sqrt{5}$. *W. T. Scott* (Evanston, Ill.).

ANALYSIS

Shenton, L. R. A determinantal expansion for a class of definite integral. I. Proc. Edinburgh Math. Soc. (2) 9, 44–52 (1953).

Let $w(x) \geq 0$ be a non-zero weight in the finite interval a, b; let $A(x)$, $B(x)$, $C(x)$ be continuous and $C(x)>0$. Writing

$$\gamma_{rs}=\int_a^b x^{r+s}w(x)C(x)dx; \quad \alpha_r=\int_a^b x^r w(x)A(x)dx,$$

$$\beta_r=\int_a^b x^r w(x)B(x)dx,$$

and denoting the matrices $(\alpha_0, \alpha_1, \cdots, \alpha_n)$, $(\beta_0, \beta_1, \cdots, \beta_n)$, and (γ_{rs}), $r, s=0, 1, \cdots, n$, by α_n, β_n, and γ_n, respectively, the author proves the following identity:

$$\int_a^b \frac{A(x)B(x)w(x)}{C(x)}dx=-\lim_{n\to\infty}\begin{vmatrix}0 & \alpha_n\\ \beta_n' & \gamma_n\end{vmatrix}:|\gamma_n|.$$

The proof is based on Parseval's formula. Various special cases are considered, in particular the case when $C(x)$ is a polynomial. *G. Szegö* (Stanford, Calif.).

Gorn, S. Maximal convergence intervals and a Gibbs type phenomenon for Newton's approximation procedure. Ann. of Math. (2) 59, 463–476 (1954).

Let $f(x)$ be twice differentiable. The Newton transform of x relative to f is (1) $Tx=x-f(x)/f'(x)$, defined wherever $f'\neq 0$. A maximum-convergence interval of T (an M-interval) is a closed interval $[M_1, M_2]$ such that the sequences of iterates T^nM_1, T^nM_2 either fail to exist or have no limit, while $\lim_{n\to\infty} T^nx$ exists for all $x \in (M_1, M_2)$. A Z-interval of T is an M-interval in which the end points M_1, M_2 are a two cycle: $TM_i=M_j$, $i\neq j$. An I-interval of f is a closed interval $[X_1, X_2]$ such that: (a) $f(x_0)=0$ for an x_0 in $X_1<x_0<X_2$; (b) $f'(x)\neq 0$ in $[X_1, X_2]$; (c) $[X_1, X_2]$ contains a unique point of inflection $\bar{x}_0$, which lies in (X_1, X_2), and at which f' has an extreme value; (d) it contains at most a finite number of solutions of $T^2x=x$; (e) $TX_1>X_2$, $TX_2<X_1$. An I-interval of T is a closed interval $[X_1, X_2]$ such that: (i) T is continuous there; (ii) $TX_1>X_2$; (iii) T is strictly decreasing from X_1 to a minimum at $\bar{x}_0$, then strictly increasing to a maximum at x_0, which is the only fixed point of T on $[X_1, X_2]$, and then strictly decreasing to TX_2 where $TX_2<X_1$; (iv) condition (d) above holds. It is seen that an I-interval for f is also one for T.

Given an I-interval for T, a detailed study (in the case $x_0\neq\bar{x}_0$) of the transformation T and its iterates is made. Among the results obtained are the following: An I-interval $[X_1, X_2]$ contains a Z-interval $[Z_1, Z_2]$. If $[y_1, y_2]\subset(Z_1, Z_2)$ then T^nx converges uniformly to x_0 on $[y_1, y_2]$. If, however, $[y_1, y_2]$ is replaced by (Z_1, Z_2), a kind of Gibbs phenomenon presents itself in the convergence of T^nx. *I. M. Sheffer.*

Karanikolov, Hr. On a formula of mechanical quadrature. Uspehi Matem. Nauk (N.S.) 9, no. 2(60), 157–161 (1954). (Russian)

The author proves that the constants $d, A_0, A_1, \cdots, A_n$ in the quadrature formula

$$\int_{-1}^{+1} f(x)dx \approx \sum_{k=1}^{n} A_k\{f(-kd)+f(kd)\}+A_0f(0)$$

may be chosen in such a way that the formula becomes exact for each polynomial of degree less than or equal to $2n+2$. The number d may be chosen as the root of the equation $\int_0^{1/d} x^2(x^2-1^2)\cdots(x^2-n^2)dx=0$ in the interval $0<d<1/n$. *G. G. Lorentz* (Detroit, Mich.).

Calculus

✶Dalton, John P. Symbolic operators. Witwatersrand University Press, Johannesburg, 1954. xvi+194 pp. £1. 10s.

The aim of this book is to provide a comparatively elementary and yet mathematically sound presentation of the Heaviside calculus. In the author's opinion Heaviside's own

theory failed for two reasons: one of these is the imperfect comprehension of the distinction between the two operators d/dt and $(\int_0^t \cdots d\theta)^{-1}$, and the other is an unfortunate choice of the primary operator on which the whole calculus is based. Removing these defects, the author is able to develop a mathematical theory of the Heaviside calculus by methods which resemble Heaviside's own methods; the theory is sufficiently general to encompass applications not only to ordinary differential equations (this has been accomplished in several different ways) but also to partial differential equations with constant coefficients. The Laplace transform theory is "an alternative and essentially different process" which "neither explains his [Heaviside's] technique nor establishes its validity." The principal difference between the two systems is that "Heaviside's basic operation is an integration over a finite time-interval, while the Laplace method starts with an integration over an infinite time-interval" and hence is forced to limit the rate of growth of the functions which it is able to handle, in a manner which is foreign to the essence of Heaviside's method.

Part I. Ordinary operators. Chapter I. Elementary operations. After some preliminary remarks, the author defines the class A of admissible functions as functions of the form $t^{a-1}f(t)$ where a is positive, and $f(t)$ is continuous on some interval $0 \leqq t \leqq T$. He defines the operators D and Q by

$$Du(t) = \frac{du}{dt}, \quad Qu(t) = \int_0^t u(\theta)d\theta$$

and sets $p = Q^{-1}$. He establishes relations between these operators, and points out that none of them are suitable for adoption as a primary operator, since D and p are not defined for all admissible functions and Q, although it is an operator on A to A, does not have an inverse defined for all admissible functions. He shows that if $\sum a_n z^n$ is a power series with a non-zero radius of convergence then $\sum a_n Q^n$ is a well-defined operator on A. Chapter II. The operators ω_λ and Ω_λ. The primary operator is $\omega_\lambda = (1-\lambda Q)^{-1}$ which is well-defined, and has an everywhere well-defined inverse, on A. The secondary operator is $\Omega_\lambda = Q\omega_\lambda$; its inverse is defined only for those admissible functions which vanish at the origin. Many properties of these operators are developed. Chapter III. The product formula

$$\prod_{\sigma=1}^{n} \frac{\omega_{\lambda_\sigma}}{\omega_{\mu_\sigma}} = \frac{f(0)}{F(0)} + \sum_{\sigma=1}^{n} \frac{f(\lambda_\sigma)}{\lambda_\sigma F'(\lambda_\sigma)} \omega_{\lambda_\sigma}$$

where $F(\lambda) = \prod_{\sigma=1}^{n} (\lambda - \lambda_\sigma)$, $f(\lambda) = \prod_{\sigma=1}^{n} (\lambda - \mu_\sigma)$, and this is equivalent to Heaviside's first expansion theorem. Moreover, under suitable assumptions about λ_σ, it is permissible to make $n \to \infty$, and this gives the extended expansion theorem which is used in connection with partial differential equations. Chapter IV. The confluent product formula. (The case of repeated factors.) Chapter V. Ordinary differential equations with constant coefficients. This chapter differs from the usual presentation only in the systematic use of ω, Ω. Chapter VI. Ordinary differential equations: examples. Chapter VII. Two-point boundary conditions. Discussion of the Green's function of $y'' - ky = 0$. Chapter VIII. Simultaneous differential equations with constant coefficients.

Part II. Partial operators of integral order. Chapter IX. Partial operators. Admissible functions are of the form $x^{a-1}t^{b-1}f(x, t)$ where a, b are positive and f is continuous in some rectangle $0 \leqq x \leqq X$, $0 \leqq t \leqq T$. The notations for the various operators are D_x, D_t, Q_x, Q_t, $\omega_\lambda = (1-\lambda Q_t)^{-1}$, $\bar{\omega}_\lambda = (1-\lambda Q_x)^{-1}$, $\Omega_\lambda = Q_t\omega_\lambda$, $\bar{\Omega}_\lambda = Q_x\bar{\omega}_\lambda$. The author derives a large number of identities and also considers operators like $\omega_{\lambda D_x} = (1-\lambda D_x Q_t)^{-1}$ and $(1-\lambda D_x \Omega_\mu)^{-1}$. Chapter X. First order partial differential equations with constant coefficients. Heaviside's unit function occurs in this chapter, and there is a brief note on impulsive functions. Apart from this, the delta function and its derivatives are carefully avoided. Chapter XI. Second order equations with constant coefficients. Mainly hyperbolic equations; Riemann's method of integration is included.

Part III: Partial operators of fractional order. Chapter XII. Fractional integrals and derivatives. Chapter XIII. The error function. Chapter XIV. Second order equations of parabolic type. Includes a thorough discussion of the "diffusion operator" e^{-xq}, $q = k^{-\frac{1}{2}}D_t^{\frac{1}{2}}$. Chapter XV. Partial differential equations under two-point boundary conditions. Examples including elliptic, hyperbolic, and parabolic equations.

There is a brief Appendix on asymptotic approximation.

The book is very clearly written, and beautifully printed. Every reader with a knowledge of elementary algebra (theory of equations) and advanced calculus, and with the mathematical maturity of an advanced undergraduate, should be able to learn from this book both the conceptual background of the author's theory of Heaviside calculus and enough manipulative technique to solve problems.

A. Erdélyi (Pasadena, Calif.).

★**Hay, G. E. Vector and tensor analysis.** Dover Publications, Inc., New York, N. Y., 1954. viii+193 pp. Unbound, \$1.50; bound, \$2.75.

After having given a definition of vectors in ordinary space and of the elementary operations, the author proceeds with some applications to analytical geometry, to differential geometry (the Serret-Frenet formulas) and to mechanics (kinematics and equations of motion of a rigid body). The chapters IV and V are devoted to differentiation and integration of vectors (Stokes, Green). The expressions for ∇f, div **a** and rot **a** are given in orthogonal curvilinear coordinates. The components of the vectors involved are supposed to be given with respect to an orthogonal system of unit vectors tangent to the parametric lines (not with respect to the natural local system of reference), which leads to rather complicated expressions. The last chapter contains an introduction to tensor analysis (metric tensor, Christoffel symbols, covariant derivative, curvature tensor). It is not clear what the author means by the remark that the components of a vector in general coordinates describe the vector "in a certain manner" and why he introduces the physical components of a vector (the orthogonal projections of the vector on the tangents of the parametric lines). By doing this the formalism of tensor theory becomes less clear.

J. Haantjes (Leiden).

Lakshmana Rao, S. K. On the evaluation of Dirichlet's integral. Amer. Math. Monthly **61**, 411–413 (1954).

Theory of Sets, Theory of Functions of Real Variables

Novotný, Miroslav. Sur un problème de la théorie des applications. Publ. Fac. Sci. Univ. Masaryk **1953**, 53–64 (1953). (Czech. Russian and French summaries)

L'auteur résout et généralise le problème suivant que O. Borůvka lui a posé [mêmes Publ. **1946**, no. 278; ces Rev. **8**, 449]: M, N étant deux ensembles, f, g étant applications

de M, resp. N, sur lui-même, déterminer toutes les applications F vérifiant $Ff(x)=gF(x)$ (le cas de Borůvka est celui où $M=N$, $f=g$). Soit β le premier ordinal dont le cardinal $>\overline{\overline{M}}, \overline{\overline{N}}$. On dira qu'un $x \varepsilon M$ possède la propriété π, $x \varepsilon (\pi)$, s'il existe une ω-suite $x_n \varepsilon M$ telle que $f(x_{n+1})=x_n$ avec $x_0=x$. Posons

$$A=\{x|x \varepsilon M, x \varepsilon (\pi)\},$$
$$B=M-A,$$
$$B_0=\{x|x \varepsilon B, f^{-1}(x)=\text{vide}\},$$
$$B_\alpha=\{x|x \varepsilon B-\bigcup B_{\alpha'}, f^{-1}(x)\subseteq \bigcup B_{\alpha'}\} \quad (\alpha'<\alpha).$$

Il existe un ordinal ϑ vérifiant $\bar{\vartheta}\leqq\beta$, $B=\bigcup B_\lambda$ $(\lambda<\vartheta)$; les B_λ sont deux à deux disjoints (L. 3). Posons: $S(x)=\beta$ si $x \varepsilon A$, $S(x)=\alpha$ si $x \varepsilon B_\alpha$. Soit R_f la partition de M en classes par la relation ρ_f où $x\rho_f y$ veut dire l'existence d'entiers m, n vérifiant $f^m(x)=f^n(y)$. L'auteur dit qu'une application F de M en N transforme f en g si $Ff(x)=gF(x)$ pour tout $x \varepsilon M$. Dans ces conditions $S(x)\leqq S(F(x))$ (L. 7) et à chaque $T \varepsilon R_f$ correspond un $T' \varepsilon R_g$ tel que $FT\subseteq T'$. L'auteur donne un procédé, K, pour construire chaque F transformant $f|T$ en $g|T'$ d'où découle la solution du problème. Deux applications en algèbre abstraite concernant les groupoïdes sont donneés. *G. Kurepa* (Zagreb).

Dwinger, Ph. On the ascending chain condition of cardinal powers of partially ordered sets. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 188–193 (1954).

This note gives simple conditions on ordered systems Y and X necessary and sufficient that the cardinal power Y^X satisfy the ascending chain condition. [For definitions see G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; these Rev. **10**, 673.] *M. M. Day* (Urbana, Ill.).

Bonferroni, Carlo. Alcune proprietà generali di un insieme variabile. Boll. Un. Mat. Ital. (3) **9**, 5–15 (1954).

Let f be a function defined on a subset of Euclidean n space E_n with values in the set of subsets of E_m. Say that L in E_m is a (stable) interlimit of $f(x)$ as $x\to a$ if $(\lim_{s\to a})$ $\liminf_{x\to a}$ dist $(L, f(x))=0$. This note discusses continuity, uniform stability, and other concepts related to the limiting behavior of such set-valued functions, and considers also results obtained in the simplified cases where each value of f is a closed or a finite set in E_m. *M. M. Day.*

Šneider, A. A. On sets appearing as a generalization of H-sets. Mat. Sbornik N.S. **34**(76), 249–258 (1954). (Russian)

A point set E situated on the real axis is said to be of type H if there are numbers d, α_k, β_k, n_k $(k=1, 2, \cdots)$ satisfying the following conditions: (1) $0<d<1$, $\beta_k-\alpha_k=d$; (2) the n_k are positive integers strictly increasing with k; (3) the intervals

$$\left(\frac{\alpha_k+i}{n_k}, \frac{\beta_k+i}{n_k}\right) \quad (i=0, \pm1, \pm2, \cdots; k=1, 2, \cdots)$$

have no points in common with E. Sets H were introduced by Rajchman in connection with problems of uniqueness of trigonometric series [see Rajchman, Fund. Math. **3**, 287–302 (1922)] and the Cantor ternary set constructed on (0, 1) and repeated periodically is of type H. The author calls a set E of type H^* if in the definition given above the integers n_k are replaced by arbitrary real numbers λ_k strictly increasing and tending to ∞ with k. The following results are established: (1) There exist sets of type H^* which are not of type H; (2) every set of type H^* consists of a finite number of sets of type H situated on non-overlapping intervals; (3) there exist two sets H situated in two non-overlapping intervals and such that the sum of these two sets is not of type H^*; (4) if a set D is of type H and $\theta>0$, then the set θD is an H^* set; and conversely every set of type H^* can be represented in the form θD, where $\theta>0$ and D is an H set. *A. Zygmund* (Chicago, Ill.).

Bauer, Heinz. Caractérisation topologique de la partie complètement additive et de la partie purement additive d'une fonction additive d'ensemble. C. R. Acad. Sci. Paris **238**, 1771–1773 (1954).

Let $\mathfrak{B}$ be a Boolean algebra of subsets of a set E which separates the points of E. Let S be a T_1-space with closure operator $\ast$ containing the set E such that: (1) E is dense in S; (2) $A\cap B=0$ in E implies $A^\ast\cap B^\ast=0$ in S; (3) every covering of S by sets in $\mathfrak{B}^\ast=\{A^\ast\}$, $A \varepsilon \mathfrak{B}$, admits a finite subcovering. Let f be a finitely additive, non-negative, real-valued measure on $\mathfrak{B}$. The resolution given by Yosida and the reviewer of f into a countably additive part f_c and a purely finitely additive part f_p [Trans. Amer. Math. Soc. **72**, 46–66 (1952); these Rev. **13**, 543] is interpreted geometrically as follows. The measure $f^\ast$ on S defined by $f^\ast(A^\ast)=f(A)$ is countably additive on $\mathfrak{B}^\ast$ and hence admits a unique extension over the σ-algebra generated by $\mathfrak{B}^\ast$, as well as an outer measure $f_e^\ast$ and an inner measure $f_i^\ast$. Then one has $f_c(A)=f_e^\ast(A)$ and $f_p(A)=f_i^\ast(A^\ast-A)$. Two corollaries are noted. No proofs are given. *E. Hewitt* (Seattle, Wash.).

Alda, Václav. On the surfaces without tangent planes. Čehoslovack. Mat. Ž. **3**(78), 154–157 (1953). (Russian. English summary)

The author points out that there is a surprising fallacy in an elementary construction used by Saks in a well known paper with the same title [Ann. of Math. (2) **34**, 114–124 (1933)]. He establishes a modification of the main statement of that paper. *L. C. Young* (Madison, Wis.).

dell'Agnola, Carlo Alberto. Sopra alcuni concetti fondamentali dell'analisi infinitesimale. Ist. Veneto Sci. Lett. Arti. Atti Cl. Sci. Mat. Nat. **110**, 65–80 (1952).

In connection with a previous note of the author [same Atti **109**, 245–260 (1951); these Rev. **13**, 730], concerning the limits of functions of subdivisions, he now discusses the existence of the Riemann and Lebesgue integrals and of the Stieltjes integral of a continuous function with respect to a function of bounded variation, and he adds some remarks concerning the functions of bounded variation and the length of a continuous curve. Throughout he applies the usual kind of reasoning. *A. Rosenthal.*

Rademacher, Hans. On the condition of Riemann integrability. Amer. Math. Monthly **61**, 1–8 (1954).

M. J. Norris has given an elementary proof of the existence of the Riemann integral of a continuous function without using the idea of uniform continuity [Amer. Math. Monthly **59**, 244–245 (1952); these Rev. **13**, 924]. This note points out that exactly the same proof was given by Gerhard Kowalewski [Grundzüge der Differential- und Integralrechnung, Teubner, Leipzig-Berlin, 1909, pp. 174–176]. In these papers the transition from local to overall properties is made by means of the mean-value theorem of the differential calculus. The author of the present paper proves in an

elementary way the following more general theorem. Let $f(x)$ be bounded on (a, b). Let $\sigma_f(x_0, h)$, $h>0$, be the oscillation of $f(x)$ on the interval (x_0-h, x_0+h). Let $\sigma_f(x_0)=\lim_{h\to 0}\sigma_f(x_0, h)$. Let S_η be the set of all points x for which $\sigma_f(x)\geqq 0$. A necessary and sufficient condition for the Riemann integrability of $f(x)$ over $[a, b]$ is that for every $\eta>0$ the set S_η can be enclosed in a finite set of intervals with total length less than η. The transition from local to overall properties is made by means of the generalized mean-value theorem for continuous functions of W. H. and G. C. Young [Quart. J. Pure Appl. Math. **40**, 1–26 (1908)].

The sequence of theorems summarized in this review is of great interest and usefulness in relation to the teaching of elementary calculus. It makes possible the complete exposition of the Riemann integral without recourse to the measure theory of Lebesgue. *R. L. Jeffery.*

Germay, R. H. Une remarque sur la théorie des fonctions intégrables. Bull. Soc. Roy. Sci. Liège **23**, 6–12 (1954).

It is proved that, if $\phi(x)$ is continuous and monotonic non-decreasing on $a\leqq x\leqq b$ and $f(x)$ is bounded, then

$$\lim \prod_i [1+M_i(\phi(x_i)-\phi(x_{i-1}))]$$

and

$$\lim \prod_i [1+m_i(\phi(x_i)-\phi(x_{i-1}))]$$

are $e^{\bar{I}fd\phi}$ and $e^{\underline{I}fd\phi}$, respectively, where $a=x_0<x_1<\cdots<x_n=b$ is any subdivision of (a, b), M_i is the sup of f on (x_{i-1}, x_i) and m_i the inf, the limit is taken as the max (x_i-x_{i-1}) tends to zero and $\bar{I}$ and $\underline{I}$ are respectively the upper and lower Stieltjes integrals of f with respect to ϕ. There are obvious corollaries for the case where f is bounded, ϕ is of bounded variation and continuous on (a, b) and the Stieltjes integral $\int_a^b fd\phi$ exists. *T. H. Hildebrandt.*

Ćetković, Simon. Sur les zéros des dérivées d'une famille des fonctions réels. Bull. Soc. Math. Phys. Macédoine **4** (1953), 25–27 (1954). (Serbo-Croatian. French summary)

Dans ce travail l'auteur part d'une famille de fonctions

$$f(x, p)=\sum_{i=1}^{n}\frac{p_i}{x-a_i},$$

$(a_i, p_i, n$ étant respectivement les nombres réels, les nombres positifs et le nombre naturel, avec $a_i<a_{i+1})$, et montre que tous les zéros réels de toutes les dérivées de la famille de fonctions $f(x, p)$ se trouvent dans l'intervalle (a_1, a_n). De même presque toutes les zéros des dérivées de la familles de fonctions $f(x, p)$ se trouvent dans un voisinage arbitrairement petit des nombres $\frac{1}{2}(a_i+a_{i+1})$ $(i=1, 2, \cdots, n-1)$. *Résumé de l'auteur.*

Bajraktarević, M. Sur les suites définies par l'équation

$$x_\nu=\epsilon_0 f(\epsilon_1 f(\cdots(\epsilon_\nu f(0))\cdots)).$$

Srpska Akad. Nauka. Zbornik Radova **35**. Mat. Inst. **3**, 61–74 (1953). (Serbo-Croatian. French summary)

Let $f(x)$ be continuous and monotonically increasing on $[-a, a]$, with $f(-a)\geqq 0$, $f(x)>x$ for $|x|<a$, and $f(a)=a$. From the binary representation $z=\sum_{\nu=0}^{\infty} d_\nu/2^\nu$ $(d_\nu=0$ or $1)$ for each $z\in I$: $[0, 2]$, define ϵ_ν $(\nu=0, 1, \cdots)$ by $\epsilon_0=1-2d_0$, $\epsilon_\nu=(1-2d_\nu)/(1-2d_{\nu-1})$, $\nu=1, 2, \cdots$, so that $\epsilon_\nu=\pm 1$.Each $z\in I$ thus determines a sequence $\{x_\nu\}$ defined in the title, and the author studies such sequences. A principal result: (A) For all $z\in I$, $\{x_\nu\}$ has at most two limit points: $\xi_1\leqq\xi_2$. (B) ξ_1, ξ_2 are monotonic functions of z, decreasing from a to $-a$, with

$$a\geqq\xi_2(z)\geqq\xi_1(z)>\xi_2(z_1)\geqq\xi_1(z_1)\geqq -a \quad (0\leqq z<z_1\leqq 2).$$

(C) $\xi_1(2-z)+\xi_2(z)=0$, $z\in I$. (D) ξ_1 is continuous on the right for $0\leqq z<2$ and ξ_2 on the left for $0<z\leqq 2$. (E) If $\xi_2>\xi_1$ for at least one point $z\in I$, then I contains a dense, denumerable set of discontinuities of ξ_1, and of ξ_2. It follows from (*E*) that for $\xi_1(z)$ to be continuous throughout I, we must have $\xi_1(z)\equiv\xi_2(z)$, so that the sequence $\{x_\nu\}$ converges for all $z\in I$. Various conditions on f, some necessary, others sufficient, are given to insure that ξ_1 is continuous on I. An illustrative example is also given. *I. M. Sheffer.*

Karamata, J. Über das asymptotische Verhalten der Folgen die durch Iteration definiert sind. Srpska Akad. Nauka. Zbornik Radova **35**. Mat. Inst. **3**, 45–60 (1953). (Serbo-Croatian. German summary)

The principal result proved is the following: Let $f(x)$ be defined in some neighborhood $0<x<c$ of $+0$, and suppose $x-f(x)$ is regular in the sense that $f(x)=x-a(x)x^kL(x)$ where as $x\to+0$, $a(x)\to a\neq 0$ and $xL'(x)=o\{L(x)\}$. Let $a>0$, $k>1$. If x_0 is chosen sufficiently small so that $f(x)>0$ for $0<x\leqq x_0$, and $\inf_{\xi\leqq x\leqq x_0}\{x-f(x)\}>0$ for all ξ in $0<\xi<x_0$, then the iterative sequence $\{x_n\}$ defined by $x_{n+1}=f(x_n)$, $n=0, 1, \cdots$, satisfies the asymptotic relation

$$x_n\sim a^*n^{-k^*}L^*(1/n) \quad (n\to\infty),$$

where $k^*=1/(k-1)$, $a^*=(k^*/a)^{k^*}$, and $x^{k^*}L^*(x)$ is the function inverse to $x^{k-1}L(x)$. An analogous result holds for $\{y_n\}$ defined by $y_{n+1}=g(y_n)$, where $g(y)$ satisfies conditions near $y=\infty$. The above result is illustrated by example. *I. M. Sheffer* (State College, Pa.).

Vituškin, A. G. Sufficient conditions for the boundedness of the linear variation of a function of three variables. Mat. Sbornik N.S. **34**(76), 307–322 (1954). (Russian)

A. S. Kronrod [Doklady Akad. Nauk SSSR (N.S.) **66**, 797–800 (1949); Uspehi Matem. Nauk (N.S.) **5**, no. 1(35), 24–134 (1950); these Rev. **11**, 19, 648] introduced and studied the notion of the linear (total) variation $V(f)$ of a continuous function $f(P)$, of a point P, defined on an interval I in two (or more) dimensions. $V(f)$ may be defined by $V(f)=\int_{-\infty}^{\infty}\Phi(t)dt$, where $\Phi(t)$ is the number of components dividing I of the level-set $[f(P)=t]$ [cf. the second paper cited, p. 66]. In the first paper [§§10 and 11, p. 799] he asserted that $f(P)$ has a finite linear variation on the n-dimensional interval I if its $(n-1)$th partial derivatives satisfy the Lipschitz condition on I. The case $n=1$ is trivial, since then $V(f)$ equals the ordinary total variation of $f(P)$ on I [cf. S. Saks, Theory of the integral, 2nd ed., Warsaw-Lwow, 1937, Chap. IX, Theorem (6.4), p. 280]. The author establishes the case $n=3$, assuming the case $n=2$. For the latter he cites Kronrod's second paper, in which the reviewer has failed to locate it: however, the reduction from $n=2$ to $n=1$ is presumably easier than that from $n=3$ to $n=2$. *H. P. Mulholland* (Birmingham).

Besicovitch, A. S. Parametric surfaces. III, I. Surfaces of minimum area. Nederl. Akad. Wetensch. Proc. Ser. A. **57**=Indagationes Math. **16**, 169–175 (1954).

[For parts I–IV see these Rev. **10**, 521; **11**, 167.] An erroneous lemma, used in the author's controversial papers, is here replaced by a series of lemmas. The correction is all the more important, since the main interest of the papers seems to the reviewer to be in the techniques employed. The

restriction to surfaces of the type of the disk, or even to surfaces of bounded topological type, was natural only as long as it was believed that this restriction would be immaterial. Such a belief has been shattered by an example anounced by Fleming [Bull. Amer. Math. Soc. 60, 367 (1954)]. Accordingly all solutions of the problem of least area, whether controversial or not, are now incomplete.

L. C. Young (Madison, Wis.).

Theory of Functions of Complex Variables

Penez, Jacqueline. Approximation by boundary values of analytic functions. Proc. Nat. Acad. Sci. U. S. A. **40**, 240–243 (1954).

Let D be a bounded domain whose frontier consists of v nonintersecting rectifiable Jordan curves. Let s denote arc length along C, and let $L_k(C)$ be the class of functions $f(z)$ defined on C for which

$$\|f\|_k=\left\{\int_C |f(z)|^k ds\right\}^{1/k}$$

exists finitely in the Lebesgue sense. Let $A_k(C)$ be the subclass of $L_k(C)$ consisting of L_k limits of functions $\phi(z)$ regular in the closure of D. Let $B_k(C)$ consist of all $f(z)\in L_k(C)$ satisfying $\int_C f(z)\phi(z)dz=0$ for such functions $\phi(z)$. Clearly $A_k(C)\subseteq B_k(C)$. The author extends a necessary and sufficient condition on C to ensure that $A_k(C)=B_k(C)$, which was obtained by Smirnov [Izvestiya Akad. Nauk SSSR. Otd. Mat. Estest. Nauk (7) **1932**, 337–372] when $v=1$, $k=2$, to the general case. She then extends many of the results on best approximations of functions in $L_k(C)$ by functions in $A_k(C)$ (or $B_k(C)$) previously obtained when D is the unit circle [W. W. Rogosinsky and H. S. Shapiro, Acta Math. **90**, 287–318 (1953); these Rev. **15**, 516, and references there given]. No proofs are given.

W. K. Hayman (Exeter).

Mandžavidze, G. F. On approximate solution of boundary problems of the theory of functions of a complex variable. Soobščeniya Akad. Nauk Gruzin. SSR **14**, 577–582 (1953). (Russian)

Let D^+ be a bounded, simply connected domain in the plane of z, bounded by a simple, closed, smooth contour L D^- is the complement of D^++L. The problem solved is that of finding a piece-wise analytic vector $\Phi(z)=(\Phi_1(z),\cdots,\Phi_n(z))$ of finite order at infinity, such that $\Phi^+=G\Phi^-+f$ on L, f and G being assigned vector and matrix, respectively, of classes $H(v)$ $(v<1)$, $H(v_1)$ $(v_1>v)$; det $G\neq0$ on L. Some of the notation and terminology is from N. I. Muskhelišvili [Singular integral equations, OGIZ, Moscow-Leningrad, 1946; these Rev. **8**, 586] and N. L. Vekua [Systems of singular integral equations . . . , Gostehizdat, Moscow-Leningrad, 1950; these Rev. **13**, 247]. Use is made of suitably defined complete, normal, linear spaces and of corresponding operators, of which

$$M\varphi=(E+G(t_0))\varphi(t_0)+\frac{1}{\pi i}(E-G(t_0))\int_L\frac{\varphi(t)dt}{t-t_0}$$

(E being the unit matrix) is typical, as well as of the theory of integral equations in the sense of principal values. Stability of partial indices is proved, as well as existence of a convergent sequence of canonical matrices.

W. J. Trjitzinsky (Urbana, Ill.).

Vekua, N. P. On a boundary problem of linear relationship. Doklady Akad. Nauk SSSR (N.S.) **94**, 173–176 (1954). (Russian)

Let D^+ be a bounded, connected domain in the plane of $z=x+iy$, its frontier L being smooth and closed; $D^-=D^++L$; origin in D^+; the angle between the tangent to L and a fixed direction is assumed to be of class H (Hölder); $\alpha(t)$ is assigned on L, $\alpha'(t)\neq0$, $\alpha'(t)\in H$; $\alpha(t)$ transforms L one-to-one on itself, reversing the direction on L. One is to find a vector $\phi^+[\alpha(t)]=G(t)\phi^+(t)+g(t)$, where matrix G (of n^2 elements) and vector g are assigned in H. The author solves (1), when $\alpha^m(t)\equiv t$, where m (>0) is an even integer and $\alpha^m(t)$ is the mth iterate of $\alpha(t)$. This constitutes an extension of results of Carleman [Verh. Internat. Math. Kongresses, Zürich, 1932, Bd. 1, Füssli, Zürich-Leipzig, 1932, pp. 138–151] when $n=1$, $m=2$, of Kveselava [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **16**, 39–80 (1948); these Rev. **14**, 152] and of the author [Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 157–180 (1952); these Rev. **14**, 153], for any n and with $m=2$.

W. J. Trjitzinsky.

Aleksandriya, G. N. On a boundary problem of linear relationship for several unknown functions. Soobščeniya Akad. Nauk Gruzin. SSR **14**, 65–70 (1953). (Russian)

Let L be a simple, closed, suitably smooth contour in the plane of $z=x+iy$, the angle between its tangent and a fixed direction being in H (Hölder class). The interior of L is denoted by S^+; $z=0$ is in S^+. Let α_1, α_2 be functions given on L, having derivatives in H distinct from zero, and transforming L one-to-one on itself, $\alpha_2(t)$ leaving the sense of direction on L unchanged and $\alpha_1(t)$ reversing it. φ is meromorphic in S^+ if φ is analytic in S^+, except possibly for a finite number of poles, and continuously extendable to be in the class H. It is proved that there always exist meromorphic vectors $\varphi_j=\{\varphi_{j1},\cdots,\varphi_{jn}\}$ $(j=1,2)$ in S^+ such that on L

$$\varphi^+_{1k}[\alpha_1(t_0)]=\sum_{i=0}^{n}\{g_1^{ki}(t_0)\varphi^+_{2i}(t_0)+g_2^{ki}(t_0)\varphi^+_{2i}[\alpha_2(t_0)]\}+g_k(t_0)\quad(k=1,\cdots,n),$$

where the coefficients are in H and det $(g_1^{ki})\neq0$ on L. This is a problem of the type originally studied by Haseman [Thesis, Göttingen, 1907]. Its solution involves use of the known theory of integral equations involving integrations in the sense of principal values, and consideration of the homogeneous adjoint problem.

W. J. Trjitzinsky.

Keldyš, M. V. On series of rational fractions. Doklady Akad. Nauk SSSR (N.S.) **94**, 377–380 (1954). (Russian)

If $\sum|A_n|<\infty$ and the function $f(z)=\sum A_n/(z-h_n)$ is meromorphic of finite order in the whole plane then (Nevanlinna's notation), as r tends to ∞, lim sup $N(r,a)/T(r)=1$ for all $a\neq0$. If $\sum A_n\neq0$, this result holds for $a=0$ also. The proof depends on various approximations for the Nevanlinna characteristics of $\sum A_n/(z-h_n)$.

A. J. Macintyre.

Lehner, Joseph. Note on the Schwarz triangle functions. Pacific J. Math. **4**, 243–249 (1954).

Put $\lambda=2\cos\pi/q$ $(q=3,4,5,\cdots)$, and define $\phi(z)$ as the function automorphic with respect to the group $\Gamma(\lambda)$ generated by $z\to z+\lambda$ and $z\to-1/z$ and with Fourier expansion

$$\phi(z)=x^{-1}+\sum_0^\infty c_n(\lambda)x^n\quad(x=\exp(2\pi iz/\lambda)).$$

The following results are proved. 1. The coefficients $c_n(\lambda)$ are

rational numbers.

2. $c_n(\lambda) \sim (2\lambda)^{-1/2} n^{-3/4} \exp 4\pi n^{1/2}/\lambda \quad (n \to \infty)$.

L. Carlitz (Durham, N. C.).

Wilson, R. Some applications of the Hurwitz-Pincherle composition theory. J. London Math. Soc. **28**, 484–490 (1953).

L'auteur précise quelque peu le théorème de composition de Hurwitz: les seules singularités possibles de $L(z) = \sum_m (\sum_r C_r^m A_{m-r} b_r) z^{-m-1}$ sont les points de la forme $a+b$ où a est une singularité de $f(z) = \sum a_n z^{-n-1}$, et où b est une singularité de $\sum b_n z^{-n-1}$. On se place dans le cas où a et b sont des pôles, ou des points singuliers isolés essentiels et on donne des conditions pour que le point $a+b$ soit de la même nature. L'ordre de $a+b$ est alors donné en fonction de ceux de a et de b (dans le cas du pôle). Son ordre, type, etc., sont donnés lorsque les points a et b sont essentiels.

S. Mandelbrojt (Paris).

Jenkins, James A. On a problem of Gronwall. Ann. of Math. (2) **59**, 490–504 (1954).

Let S be the class of functions $f(z) = z + a_2 z^2 + \cdots$ regular and univalent in $|z| < 1$, and normalized so that $a_2 \geqq 0$. Let $S(c)$ denote the subclass of S for which $a_2 = c$. The author solves a classical problem by constructing for every pair r, c in the ranges $0 < r < 1$, $0 \leqq c \leqq 2$ a function $f(z) = F(z, r, c) \in S(c)$ which gives to $M(r, f) = \sup_{|z|=r} |f(z)|$ its maximum value $b = b(r, c)$ for $f(z) \in S(c)$.

The following features of the solution are of interest. (i) $w = F(z, r, c)$ satisfies the differential equation

$$\left(\frac{dw}{dz}\right)^2 = \frac{w^2(1-w/b)[(z-e^{i\theta})(z-e^{-i\theta})]^2}{z^2(z-r)(z-1/r)(1-w/a)},$$

where $a > b$ or $a < 0$ and $z = -1, 0, r$ correspond to $w = a, 0, b = b(r, c)$. The constant θ, $0 \leqq \theta \leqq \pi$ can then be uniquely chosen so that $F(z, r, c) \in S(c)$. (ii) Hence $F(z, r, c)$ has real coefficients and attains its maximum modulus on $|z| = r$ at $z = r$. (iii) In the second half of the paper equations for $b(r, c)$ involving elliptic functions are obtained. From these the author deduces that, for fixed c,

$$(1-r)^2 b(r, c) \to 4a_0 \exp\{2 - 4(a_0)^{1/2}\}$$

as $r \to 1$, where $a_0 = [2 - (2-c)^{1/2}]^{-2}$. (iv) The reviewer has recently shown that, for $f(z) \in S$, $(1-r)^2 r^{-1} M(r, f)$ decreases with r and that the limits

$$\alpha = \lim_{r\to 1} (1-r)^2 M(r, f) = \lim_{n\to\infty} |a_n|/n$$

exist and are equal [C. R. Acad. Sci. Paris **237**, 1624–1625 (1953); these Rev. **15**, 516]. Hence the exact upper bound for α, when $f(z) \in S(c)$ is $4a_0 \exp\{2 - 4(a_0)^{1/2}\}$.

The proof uses the methods of extremal lengths or module problems and symmetrization in a way the author has successfully applied them to a variety of other problems [notably Amer. J. Math. **75**, 510–522 (1953); these Rev. **15**, 115]. Here there are in addition some very delicate arguments involving orders of magnitude. *W. K. Hayman.*

Robertson, M. S. Schlicht solutions of $W'' + pW = 0$. Trans. Amer. Math. Soc. **76**, 254–274 (1954).

For earlier work see Z. Nehari [Bull. Amer. Math. Soc. **55**, 545–551 (1949); these Rev. **10**, 696] and E. Hille [ibid. **55**, 552–553 (1949); these Rev. **10**, 696]. Rather more general than the title suggests, the paper considers the univalence of

$$F(z) = \{W(z)\}^\lambda = z + \cdots,$$

where $W''(z) + p(z)W(z) = 0$. The paper considers first special functions $p(z)$ and develops a sort of subordination principle from which theorems for general $p(z)$ are associated with the corresponding special case. The main theory needs too much space for statement here. It is illustrated by five examples, the first being: If $zp(z)$ is regular and $\Re\{z^2 p(z)\} \leqq \frac{1}{4} X_1^2 |z|$ for $|z| < 1$, then the solution $W = z + a_2 z^2 + \cdots$ of $W'' + pW = 0$ is schlicht and starlike in $|z| < 1$. X_1 is the smallest positive zero of $J_0(X)$ and $\frac{1}{4} X_1^2$ is the best constant. *A. J. Macintyre.*

Nehari, Zeev, and Schwarz, Binyamin. On the coefficients of univalent Laurent series. Proc. Amer. Math. Soc. **5**, 212–217 (1954).

Let $\sum_{-\infty}^{\infty} a_n z^n$ converge for $\rho < |z| < 1$ to a univalent function $f(z)$ representing this annulus on a domain D. If the coefficients a_n are all real, then

$$|a_n| \leqq \frac{n}{1-\rho^{2n}} |a_1(1+\rho^{n+1}) - a_{-1}(1-\rho^{n-1})|$$

for $n = 2, 3, \cdots$ with a similar inequality for $n = -2, -3, \cdots$. Equality is attained by certain examples related to elliptic functions. If $a_0 = 0$ and D is "starlike" with respect to the origin, then

$$|a_n| \leqq \frac{n}{1-\rho^{2n}} [|a_1|(1+\rho^{n+1}) + |a_{-1}|(1+\rho^{n-1})].$$

It is conjectured that this inequality is not best possible.

A. J. Macintyre (Aberdeen).

Gel'fer, S. A. On coefficients of typically real functions. Doklady Akad. Nauk SSSR (N.S.) **94**, 373–376 (1954). (Russian)

Let $F(z) = \sum_{n=-1}^{\infty} \alpha_n z^n$ be typically-real in $0 < |z| < 1$. It is proved that (a) if $\alpha_1 - \alpha_{-1} = 1$ then $|\alpha_n| \leqq n$, $n \geqq 2$, and (b) if instead $\alpha_{-1} = -1$ and $F(z) \neq 0$ in $|z| < 1$ then $-2 \leqq \alpha_0 \leqq 2$, $-1 \leqq \alpha_1 \leqq 3$, and

$$4 \min_{0\leqq\theta\leqq\pi} \sin n\theta \sin\theta \leqq \alpha_n \leqq 4 \max_{0\leqq\theta\leqq\pi} \sin n\theta \sin\theta \quad \text{for} \quad n \geqq 2.$$

All the inequalities are sharp. In the proof of (b) the author uses a Stieltjes integral representation which he attributes to Goluzin, but which was proved earlier by M. S. Robertson [Bull. Amer. Math. Soc. **41**, 565–572 (1935)]. The result (a) is partly contained in a more general theorem of Nehari and Schwarz [see the preceding review].

A. W. Goodman (Lexington, Ky.).

Singh, S. K. The maximum term and the rank of an entire function. Publ. Math. Debrecen **3** (1953), 1–8 (1954).

Let $f(z)$ be an entire function, $\mu(r)$ the maximum term in its power series, $\nu(r)$ the rank of this term; $M(r)$, $n(r)$ and $N(r)$ have their usual meanings. The following results are established. (1) If $f(z)$ is of finite order ρ and $\log x = o(\Phi(x))$ then $\nu(r)\Phi(r)/\log M(r) \to \infty$; and if $\rho > 0$, the same quantity is unbounded for any $\Phi(r) \to \infty$. (2) If a is an exceptional value of $f(z)$, then $n(r, x)\Phi(r)/\log M(r) \to \infty$, $x \neq a$, for any $\Phi(r) \to \infty$. (3) If $f(z)$ is a canonical product of integral order ρ and genus ρ and if $\inf n(r)\Phi(r) r^{-\rho-1} > 0$ then

$$\limsup n(r)\Phi(r)/\log M(r) = \infty.$$

(4) For large r, $\mu'(r) \geqq r^{-1}\mu(r)\{\log \mu(r)\}/\log r$. [Here $\mu'(r)$ is the derivative of $\mu(r)$, whereas in Vijayaraghavan's theorem in which M replaces μ, $M'(r)$ is the maximum modulus of $f'(z)$; hence the two theorems are not as much alike as the author seems to think.] (5) The integrals

$\int^\infty t^{-m-1} \log \mu(t)dt$ and $\int^\infty t^{-m-1}\nu(t)dt$ converge or diverge together [the hypothesis $m>0$ being apparently tacitly assumed]. (6) If $f(z)$ has no zeros for $|z|<1$, then $M(R)/m(r) \geqq (R/r)^{N(r)/\log R}$, $0<r<R$.

R. P. Boas, Jr. (Evanston, Ill.).

Myrberg, P. J. Über die Picardsche Gruppe. Rend. Circ. Mat. Palermo (2) 2 (1953), 169–176 (1954).

Let G denote the Picard group consisting of the linear fractional transformations $z'=(\alpha z+\beta)/(\gamma z+\delta)$, where α, β, γ, δ are Gaussian integers satisfying $\alpha\delta-\beta\gamma=1$. Let K denote a circle in the z-plane and let $G(K)$ denote the family of limiting sets (Häufungsgebilde) of $\{TK|T \in G\}$. The following theorems are established. I. Almost all circles K have the property that $G(K)$ is the set of all circles of the plane. II. There exist K such that $G(K)$ consists of null circles. Every circle of the plane is a cluster circle of the family of these exceptional circles. Several corollaries are given. *M. Heins* (Providence, R. I.).

Mullender, P. On some conformal mappings. Simon Stevin 30, 44–47 (1954). (Dutch)

It was shown in an earlier note [Simon Stevin 26, 136–142 (1949); these Rev. 10, 697] that for a Remak series, $w(z)=z-\sum_{n=2}^\infty p_n z^n$, the image of $|z|\leqq 1$ will have a cusp at $w(1)$, if $\sum_{n=1}^\infty q_n$ is convergent, where $q_n=\sum_{\nu=1}^\infty \nu p_{n+\nu}$. The author now shows by an example that this sufficient condition for a cusp is not a necessary one. *A. W. Goodman.*

Heins, Maurice. Studies in the conformal mapping of Riemann surfaces. II. Proc. Nat. Acad. Sci. U. S. A. 40, 302–305 (1954).

Continuation of part I [same Proc. 39, 322–324 (1953); these Rev. 14, 862], containing new results (without proofs) on the following topics: (1) properties of the quasi-bounded component v_q of the residual term u_q [loc. cit.]; (2) mappings of non-negative harmonic functions; (3) a subclass of asymptotic points distinguished by metric properties; (4) properties of maps which are locally of type Bl; in particular, sufficient conditions that a map be of this type.

L. Sario (Cambridge, Mass.).

Graeub, W. Über die schwächste Uniformisierende. Math. Z. 60, 66–78 (1954).

The classical realization problem of uniformization theory to construct an "analytische Gebilde" conformally equivalent to a given abstract Riemann surface is treated. No reference is made to the well-known work on the subject [e.g. Koebe, C. R. Acad. Sci. Paris 148, 1446–1448 (1909)] nor do the results go as far as those of Koebe.

M. Heins (Providence, R. I.).

Potyagaĭlo, D. B. Condition of hyperbolicity of a class of Riemannian surfaces. Ukrain. Mat. Žurnal 5, 459–463 (1953). (Russian)

Let F be a Riemann surface obtained as follows. The w-plane ($w=u+iv$) is slit along the lines $u<0$, $v=v_k$ and $u<0$, $v=v_k'$, where $k=1, 2, \cdots$ and $v_k>0$, $v_k'<0$, to yield a sheet H_0. To H_0 is attached a sheet H_k along the slit $u<0$, $v=v_k$ and a sheet H_k' along the slit $u<0$, $v=v_k'$; H_k and H_k' each have the structure of the w-plane minus one slit. The resulting surface F is simply connected and has branch points of first order over $v_k i$ and $v_k' i$. It is proved that if $v_k=k$ and $|v'_{k+1}|/|v_k'|\geqq q^k$, $q>1$, then F is hyperbolic. The proof is based on quasi-conformal mapping and on results of Volkovyskiĭ [Trudy Mat. Inst. Steklov. 34 (1950); these Rev. 14, 156]. *W. Kaplan* (Ann Arbor, Mich.).

Goldman, Oscar. Analytic almost-periodic functions. I. Proc. Nat. Acad. Sci. U. S. A. 40, 294–296 (1954).

The author considers the group R of positive numbers, a compact, connected Abelian group G, and a continuous homomorphism h of R into G such that $h(R)$ is dense in G. For $z=x+iy$ he puts $e(z)=[\exp x, h(\exp y)]$, and a function f defined on an open set U in $R\times G$ is called analytic if, for every $\alpha \in R\times G$, the function $f[\alpha e(z)]$ is analytic in $e^{-1}(\alpha^{-1}U)$. The function f is analytic if and only if the two derivatives

$$f_1'(r, g)=\frac{d}{dx}f[re^x, g]_{x=0}, \quad f_2'(r, g)=\frac{d}{dy}f[r, gh(e^y)]_{y=0}$$

are continuous and $f_1'+if_2'=0$. The author further states a theorem which is a generalization of Cauchy's theorem for ordinary analytic functions. *H. Tornehave.*

Theory of Series

Tevzadze, N. R. Summation of double numerical series by the method of Lebesgue. Soobščeniya Akad. Nauk Gruzin. SSR 14, 71–76 (1953). (Russian)

A double series $\sum a_{m,n}$ is evaluable to S by the Lebesgue method if $\lim_{u,v\to 0}F(u, v)=S$, where

$$F(u, v)=\sum_{m,n=1}^\infty a_{mn}\frac{\sin mu}{mu}\frac{\sin nv}{nv}.$$

It is shown that if the series in

$$A_m=m\sum_{k=1}^\infty |a_{mk}|, \quad B_n=n\sum_{k=1}^\infty |a_{kn}|$$

are all convergent and if $\lim A_m=0$ and $\lim B_n=0$, then $\sum a_{mn}$ is evaluable to S by the Lebesgue method if and only if $\sum a_{mn}$ converges to S. *R. P. Agnew* (Ithaca, N. Y.).

Berekašvili, V. A. Borel summation of double series. Soobščeniya Akad. Nauk Gruzin. SSR 14, 193–196 (1953). (Russian)

If a double series $\sum a_{m,n}$ with partial sums $S_{m,n}$ converges to S, and if positive constants A and a and bounded sequences B_n and C_n exist such that

$$|S_{m,n}|\leqq Aa^{B_m+C_n},$$

then $\sum a_{m,n}$ is restrictedly evaluable to S by the Borel exponential method for which the transform of $\sum a_{m,n}$ is

$$e^{-(t+u)}\sum_{m,n=0}^\infty S_{mn}\frac{t^m}{m!}\frac{u^n}{n!}.$$

R. P. Agnew (Ithaca, N. Y.).

Sunouchi, Gen-ichirô. Corrections to the paper "Tauberian theorems for Riemann summability." Tôhoku Math. J. (2) 5, 189 (1953).

See same vol. 34–42 (1953); these Rev. 15, 304.

Reid, William T. A Tauberian theorem for power series. Math. Z. 60, 94–97 (1954).

A Tauberian theorem is proved by modifying the proof of Wielandt [Math. Z. 56, 206–207 (1952); these Rev. 14, 265] of the fact that, if $\sum a_n$ is a series of real terms which is evaluable by the Abel method and such that $na_n\leqq K$, then $\sum a_n$ is convergent. A sequence $d_0, d_1, d_2, \cdots$ is said to satisfy conditions Γ if (i) $d_n\geqq 0$, (ii) $\sum d_n x^n$ converges

when $|x|<1$, (iii) there is a constant M for which $g(x)=(1-x)\sum d_n x^n \leq M$ when $0\leq x<1$, and (iv) for each positive integer, k, $g(x^k)-g(x)\to 0$ as $x\to 1$. It is shown that if $\sum a_n$ is evaluable by the Abel method and if there is a sequence d_n satisfying conditions Γ for which $na_n \leq d_n$ then $\sum a_n$ is convergent. The conditions Γ are clearly satisfied by each sequence d_n of nonnegative numbers which is evaluable by the Abel method and hence by each sequence d_n of nonnegative numbers evaluable C_1. It is not difficult to answer the author's question whether there exists a sequence d_n that satisfies condition Γ but is not evaluable by the Abel method. An example is furnished by each bounded sequence d_n of real nonnegative numbers for which the Abel transform diverges and oscillates sufficiently slowly and hence a fortiori by each bounded divergent sequence d_n of real nonnegative numbers which oscillates sufficiently slowly. *R. P. Agnew* (Ithaca, N. Y.).

Frank, Evelyn, and Perron, Oskar. Remark on a certain class of continued fractions. Proc. Amer. Math. Soc. **5**, 270–283 (1954).

In a certain collection of continued fraction expansions for the geometric series $\sum z^p$, there is one which converges to $1/(1-z)$, one which converges to a meromorphic function of $1/z$ having 0 as essential singularity, and one which converges to $1/(1-z)$ for $|z|<\frac{1}{2}$ and to $4-1/z$ for $|z|>\frac{1}{2}$. *H. S. Wall* (Austin, Tex.).

Fourier Series and Generalizations, Integral Transforms

Bari, N. K. Generalization of inequalities of S. N. Bernštein and A. A. Markov. Izvestiya Akad. Nauk SSSR. Ser. Mat. **18**, 159–176 (1954). (Russian)

The paper supplies proofs of results announced previously without proof [Doklady Akad. Nauk SSSR (N.S.) **90**, 701–702 (1953); these Rev. **15**, 215]. To the results mentioned in the review of the earlier paper (loc. cit.) the following one should be added. Let $T(x)$ be a trigonometric polynomial of order n, (a, b) a subinterval of $(0, 2\pi)$ and $1\leq p<q\leq\infty$. Then

$$\left(\int_a^b |T|^q dx\right)^{1/q} \leq Cn^2(p^{-1}-q^{-1})\left(\int_a^b |T|^p dx\right)^{1/p},$$

where C depends on $b-a$ only. *A. Zygmund.*

Helson, Henry. Proof of a conjecture of Steinhaus. Proc. Nat. Acad. Sci. U. S. A. **40**, 205–206 (1954).

The author proves the following conjecture of Steinhaus. Let $\sum_{k=-\infty}^{\infty} a_k e^{ikx}$ be a trigonometric series with the property that the partial sums $\sum_{k=-N}^{N} a_k e^{ikx}$ are all non-negative for all x; then $|a_k|$ tends to 0 as $|k|$ tends to infinity. In fact, the author proves a stronger theorem; he assumes only that

$$\int_0^{2\pi}\left|\sum_{k=-N}^{N} a_k e^{ikx}\right| dx$$

is bounded. *P. Erdös.*

Lorch, Lee. The principal term in the asymptotic expansion of the Lebesgue constants. Amer. Math. Monthly **61**, 245–249 (1954).

The author gives a new simple proof of Fejér's theorem that the nth Lebesgue constant is $4\pi^{-2}\log n+O(1)$ and some allied theorems [cf. Hardy and Rogosinski, Fourier series, Cambridge, 1944, p. 54; these Rev. **5**, 261; Zygmund, Trigonometrical series, Warszawa-Lwów, 1935, p. 172]. *S. Izumi* (Tokyo).

Chow, Hung Ching. An extension of a theorem of Zygmund and its application. J. London Math. Soc. **29**, 189–198 (1954).

Let $f(z)$ of class H^p, $0<p\leq 2$, have the representation $f(z)=\sum_{n=0}^{\infty} c_n r^n e^{ni\theta}$, $0\leq r<1$, and let $t_n^{\alpha}(\theta)$ be the (C, α) mean corresponding to $t_n^0=nc_n e^{ni\theta}$. The author shows that $\sum_{n=0}^{\infty}|t_n^{\alpha}(\theta)|^2 n^{-1}$ is convergent for almost all θ, where $\alpha=1/p$ or $\alpha>1/p$ according as $0<p\leq 1$ or $1<p\leq 2$. For $p=1$ the result had been established by Zygmund [Fund. Math. **30**, 170–196 (1938)]. As a consequence it is shown that $\{\log(n+1)\}^{-\frac{1}{2}-\delta}$, $\delta>0$, is a summability factor for $|C, \alpha|$ summability of $\sum_{n=0}^{\infty} c_n e^{nt}$ with α and p related as above. For $p=1$, this result had been obtained earlier by the author [J. London Math. Soc. **16**, 215–220 (1941); these Rev. **4**, 37]. *P. Civin* (Eugene, Ore.).

Yano, Shigeki. A remark on absolute Cesàro summability of Fourier series. Tôhoku Math. J. (2) **5**, 194–195 (1953).

It is shown that for $p\geq 1$, the summability $|C, 1/p|$ of the Fourier series of functions of $L^p(0, 2\pi)$ is not a local property. For $p=1$ this result was due to Bosanquet and Kestelman [Proc. London Math. Soc. (2) **45**, 88–97 (1939)]. For $1<p\leq 2$ Tsuchikura [Tôhoku Math. J. (2) **5**, 52–66 (1953); these Rev. **15**, 417] has shown $|C, r|$ summability is a local property for the Fourier series of $L^p(0, 2\pi)$ for $r>1/p$. *P. Civin* (Eugene, Ore.).

Yano, Shigeki. Cesàro summability of Fourier series. Tôhoku Math. J. (2) **5**, 196–197 (1953).

Let $f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$. The author improves his earlier result [Pacific J. Math. **2**, 419–429 (1952); J. Math. Tokyo **1**, 32–34 (1951); these Rev. **14**, 267, 552] by showing that for $1\leq p\leq 2$ and $f(x)\in L^p$, $0<\alpha<1$, the series $\sum_{n=1}^{\infty} n^{-\alpha/p}(a_n\cos nx+b_n\sin nx)$ is summable $(C, -\alpha/p)$ except on a set of $(1-\alpha)$-capacity zero. For $p>2$ the best possible statement concerning the exceptional set is that it is of β-capacity zero for $\beta>1-\alpha$. *P. Civin* (Eugene, Ore.).

Sunouchi, Gen-ichirô. Cesàro summability of Fourier series. Tôhoku Math. J. (2) **5**, 198–210 (1953).

Let $\varphi(t)\sim\sum_{n=1}^{\infty} a_n\cos nt$, and let

$$\Phi_\alpha(t)=\frac{1}{\Gamma(\alpha)}\int_0^t \varphi(u)(1-u)^{\alpha-1}du \quad (\alpha>0).$$

The author establishes his earlier conjecture that if $0<\beta<\gamma$ and $\Phi_\beta(t)=o(t^\gamma)$ as $t\to 0$ then the Fourier series of $\varphi(t)$ is summable $(C, \beta/(\gamma-\beta+1))$ to zero at $t=0$. In the opposite direction he establishes that if s_n^β denotes the nth Cesàro mean of order β, $s_n^\beta=o(n^\gamma)$ as $n\to\infty$ for $\beta>\gamma>-1$, $1+\alpha>\delta$ and $\sum_{\gamma=n}^{\infty}|a_\gamma|\gamma^{-1}=O(n^{-1+\delta})$ as $n\to\infty$, then $\Phi_\alpha(t)=o(t^\alpha)$ for $\alpha=\delta(\beta+1)/(\beta-\gamma+\delta)$. Weaker results in each direction had been obtained earlier by several authors. *P. Civin.*

Alexits, Georges. Sur la sommabilité des séries orthogonales. Acta Math. Acad. Sci. Hungar. **4**, 181–189 (1953). (Russian summary)

Let $\{\phi_n(x)\}$ be an orthonormal system over $\langle a, b\rangle$ and let the $L_n(x)$ be its Lebesgue constants. Theorem: If the $L_{2^n}(x)$ are uniformly bounded on a subset E of $\langle a, b\rangle$, and if $\sum|c_n|^2<\infty$, then $\sum c_n\phi_n(x)$ is summable (C, α) p.p. on E,

for every $\alpha > 0$. This result is applied to the systems of Walsh and Haar. An earlier theorem of Kaczmarz [Studia Math. 1, 87–121 (1929)], in which the $L_{2^n}(x)$ are replaced by the $L_n^{(1)}(x)$, the Lebesgue constants of the $(C, 1)$-method, is also reproved. *W. W. Rogosinski.*

Lewis, D. C. Orthogonal functions whose derivatives are also orthogonal. Rend. Circ. Mat. Palermo (2) **2** (1953), 159–168 (1954).

Let $\{y_n(x)\}$ be orthogonal with the weight $w(x)$ in (a, b), let $y_n(x)$, $p(x)$, $q(x)$ be of class C'' and let us assume that $\{q(py_n)'\}$ is an orthogonal system with the weight $v(x)$, $v(x)$ in C'. Moreover, let

$$\int_a^b [q(py_n)']^2 dx \Big/ \int_a^b y_n^2 w dx = \lambda_n.$$

Then $y_n(x)$ satisfies the differential equation

$$\frac{d}{dx}[p^2q^2vy_n'] + [p(p'q^2v)' + \lambda_n w(x)]y_n = 0$$

provided the boundary conditions

$$p^2q^2vy_ny_m' + pp'q^2vy_ny_m = 0 \quad \text{for} \quad x = a \quad \text{and} \quad x = b$$

are satisfied. Another result is in a certain sense the converse of the above theorem. The important case of Sturm-Liouville equations is considered and special cases are treated. *G. Szegő* (Stanford, Calif.).

Lukacs, Eugene, and Szász, Otto. Nonnegative trigonometric polynomials and certain rational characteristic functions. J. Research Nat. Bur. Standards **52**, 153–160 (1954).

The authors consider a rational function $\phi(t)$ proportional to $z^{-1}\prod_{k=1}^m (z^2 - d_k^2)/\prod_{j=1}^n (z^2 - b_j^2)$, where $z = t + ia$, and $a > 0$, $0 < b_1 < \cdots < b_n$, $0 < d_1 < \cdots < d_m$, $m \leqq n$. They remark that $\phi(t)$ is then the characteristic function of a (probability) distribution if and only if ($\phi(0) = 1$ and)

$$(*) \quad g(\theta) = \begin{vmatrix} 1-\lambda_1 \cos b_1\theta & 1-\lambda_2 \cos b_2\theta & \cdots & 1-\lambda_n \cos b_n\theta \\ b_1^2 & b_2^2 & \cdots & b_n^2 \\ \cdot & \cdot & \cdots & \cdot \\ b_1^{2n-2} & b_2^{2n-2} & \cdots & b_n^{2n-2} \end{vmatrix} \geqq 0$$

for all real θ, where $\lambda_j = \prod_{k=1}^m (1 - b_j^2 d_k^{-2})$ $(j = 1, \cdots, n)$; and they proceed to investigate the conditions under which (*) holds in certain special cases, in all of which the numbers b_j are positive integers and $g(\theta)$ is thus a trigonometric polynomial. They considered the case where $m = 0$ (and thus $\lambda_j = 1$) in a previous paper [same J. **48**, 139–146 (1952); these Rev. **14**, 161]. Now for $m = 1$ with (i) $b_j = j$ $(j = 1, \cdots, n)$ they show that (*) holds if and only if $2d_1^2 \geqq n$, but with (ii) $b_j = 2j$ it holds if and only if $d_1^2 \geqq 2n$. They settle also the case where $m = 1$ and (iii) $b_j = 2j - 1$, and the cases where $n = 2$ and b_j is given by (i), (ii), or (iii), but the results for these cases are too complicated to be given here. *H. P. Mulholland* (Birmingham).

Lukacs, Eugene, and Szász, Otto. Certain Fourier transforms of distributions. II. Canadian J. Math. **6**, 186–189 (1954).

Complétant deux précédents mémoires [même J. **3**, 140–144 (1951); Pacific J. Math. **2**, 615–625 (1952); ces Rev. **12**, 823; **14**, 485] et un résultat de Takano [Tôhoku Math. J. (2) **3**, 306–315 (1951); ces Rev. **13**, 937], les auteurs indiquent des conditions nécessaires auxquelles doit satisfaire une fraction rationnelle $\phi(t)$ pour être une fonction caractéristique; par exemple, si les seuls poles de $\phi(t)$ de partie imaginaire a $(a > 0)$ sont ia et $\pm b + ia$, et si ces poles sont d'ordre s, il faut que $A_{1,s}a^s - 2|C_{1,s}|(a^2 + b^2)^{s/2} \geqq 0$, où $A_{1,s}$ est le coefficient de $(1 - it/a)^{-s}$ et $C_{1,s}$ celui de $(1 - it/a + ib)^{-s}$ dans la décomposition de $\phi(t)$ en éléments simples. Il serait trop long de reproduire en entier l'énoncé des auteurs; il peut s'étendre à des cas où $\phi(t)$ est une fonction méromorphe. *R. Fortet* (Paris).

Saksena, K. M. Generalizations of Stieltjes transform. Bull. Calcutta Math. Soc. **45**, 101–107 (1953).

The Stieltjes transform may be envisaged as an iterated Laplace transform. The author combines the Laplace transform with Varma's generalized Laplace transform [Current Sci. **16**, 17–18 (1947); these Rev. **9**, 346] and similar transforms, and also these transforms with one another, to obtain formally, five generalizations of the Stieltjes transform. The simplest of the five is

$$f(s) = \frac{\Gamma(a)}{s\Gamma(c)} \int_0^\infty {}_2F_1(a, 1; c; -t/s)\psi(t)dt.$$

[$a = c$ is the Stieltjes transform.] *A. Erdélyi.*

Wuyts, P. On the convergence-abscissa of a Laplace integral. Nieuw Arch. Wiskunde (3) **2**, 1–27 (1954).

The author derives a number of formulas, generalizing results of Malmrot, Fujiwara and Knopp for the abcissa of convergence β of the Laplace transform: $\int_0^\infty \exp(-su)F(u)du$. Thus, it is shown that if $q(t)$ is an increasing function satisfying certain conditions then:

$$\beta = \limsup_{t\to\infty} \left(\sup_{h \in (t, q)} t^{-1} \log \left| \int_h^t F(u)du \right| \right).$$

This includes Knopp's result [Math. Z. **54**, 291–296 (1951); these Rev. **13**, 127]. *S. Agmon* (Jerusalem).

***Doetsch, Gustav. L'application de la transformation bidimensionnelle de Laplace dans la théorie des équations aux dérivées partielles.** Premier colloque sur les équations aux dérivées partielles, Louvain, 1953. pp. 63–78. Georges Thone, Liège; Masson & Cie, Paris, 1954.

In this expository paper the author shows that the solution of partial differential equations by means of the (double) Laplace transformation often involves too many boundary data in the first place; the superfluous boundary data can usually be eliminated by the condition that the operational solution be a Laplace transform. The material presented is that of section 9 and section 10 of Voelker and Doetsch, Die zweidimensionale Laplace-Transformation [Birkhäuser, Basel, 1950; these Rev. **12**, 699]. *A. Erdélyi* (Pasadena, Calif.).

Berkeš, Branko. Fouriersche Reihen und Laplacesche Transformation. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **8**, 196–212 (1953). (Serbo-Croatian summary)

The most valuable part of this paper is a table giving the graphs, operational images, and Fourier series of 66 periodic functions. There are also ten pages of expository material on Laplace transforms of periodic functions. *A. Erdélyi* (Pasadena, Calif.).

Bose, N. N. On certain formulae in operational calculus. Bull. Calcutta Math. Soc. **45**, 95–100 (1953).

If $\phi(p)$ is the operational image of $f(t)$, then the operational image of $t^m\phi(t^{-1})$ is

$$4p^{1-m/2}\int_0^\infty t^{m+1}f(t^2)K_m(2p^{\frac{1}{2}}t)dt$$

and

$$\int_0^\infty x^{2n}e^{x^2}f(x^2)D_{-2n}(2x)dx$$

$$=\frac{(-1)^n}{\pi^{\frac{1}{2}}2^{n+1}}\int_0^\infty v^{n-\frac{1}{2}}{}_2F_1(n+\tfrac{1}{2}, n; \tfrac{1}{2}; -2v)\phi(v^{-1})dv,$$

$$\int_0^\infty x^{2n+1}e^{x^2}f(x^2)D_{-2n-3}(2x)dx$$

$$=\frac{(-1)^n}{\pi^{\frac{1}{2}}2^{n+1}}\int_0^\infty v^{n+\frac{1}{2}}{}_2F_1(n+\tfrac{3}{2}, n+2; \tfrac{3}{2}; -2v)\phi(v^{-1})dv.$$

The author gives a formal derivation of these relations together with numerous examples and applications.

A. Erdélyi (Pasadena, Calif.).

Hull, T. E., and Wolfe, W. A. On inverting Laplace transforms of the form $h(s)/(p(s)+q(s)e^{-rs})$. Canadian J. Physics **32**, 72–80 (1954).

Some physical problems, especially those related to difference-differential equations, have a function $U(t)$ of time t as solution which gives rise to a Laplace transform $u(s)=h(s)\cdot(p(s)+q(s)e^{-as})^{-1}$ with $q(s)$ and $p(s)$ as polynomials in s and with $a>0$. Under the assumption that $q(s)$ has lower degree than $p(s)=s^n+p_1s^{n-1}+\cdots+p_n$ the authors develop a Volterra integral equation

$$U(t)=F(t)-\int_0^t G(x)U(t-x)dx$$

for $U(t)$, which is to be solved by iteration. An essential feature of this equation is, that $G(x)=0$ for $x>a$. For small values of $a>0$ the integral in the equation represents merely a correction term, and the iterations converge rapidly. In some cases taken from the theory of counters and of heat transfer there is another advantage: The iteration leads to lower and upper bounds for the solution. In order to obtain a $G(x)$ vanishing for $x>a$, n roots of the equation $p(s)+q(s)e^{-as}=0$ are determined in such a way that they approach the zeros of $p(s)$ when $a\to 0$. Let $s_1, s_2, \cdots, s_n$ be these roots, which are assumed to be distinct. With $r(s)=\prod_{k=1}^n(s-s_k)$ a suitable $G(x)$ can be found from its Laplace transform

$$g(s)=\sum_{k=1}^n\frac{p(s_k)+q(s_k)e^{-as}}{r'(s_k)(s-s_k)}$$

and $F(t)$ from its Laplace transform $f(s)=h(s)/r(s)$.

H. Bückner (Schenectady, N. Y.).

Bhatnagar, K. P. Two theorems on self-reciprocal functions and a new transform. Bull. Calcutta Math. Soc. **45**, 109–112 (1953).

This is a continuation of two earlier papers by the same author [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 42–69 (1953); Ganita **4**, 19–37 (1953); these Rev. **14**, 977; **15**, 216]. The author gives a characterization of R_μ functions, and also certain operators which transform R_μ functions into other R_μ functions, also extensions to $R_{\mu,\lambda,\ldots}$.

A. Erdélyi (Pasadena, Calif.).

Delerue, Paul. Sur l'application du calcul symbolique à deux variables aux calcul d'intégrales simples. C. R. Acad. Sci. Paris **238**, 1686–1688 (1954).

If $f(p, q)$ is the two-dimensional Laplace transform of $F(x, y)$, then $f(p, p)$ is the one-dimensional Laplace transform of (1) $\int_0^x F(x-s, s)ds$. The author gives several examples to show how this circumstance can be used to evaluate integrals of the form (1).

A. Erdélyi.

Polynomials, Polynomial Approximations

Yakovkin, M. V. Necessary and sufficient conditions for reducibility of polynomials. Doklady Akad. Nauk SSSR (N.S.) **93**, 629–631 (1953).

The author proves the theorem: Let

$$f(x)=\sum_{k=0}^n a_kx^{n-k},\quad \varphi(x)=\sum_{k=0}^p b_kx^{p-k},\quad \psi(x)=\sum_{k=0}^q c_kx^{q-k}$$

be polynomials with integral non-negative coefficients. Then in order that $f(x)=\varphi(x)\psi(x)$ it is necessary and sufficient that $f(1)=\varphi(1)\psi(1)$ and that $f(t)=\varphi(t)\psi(t)$ for some integer t exceeding all the coefficients of the polynomials. This is used to formulate criteria for reducibility of polynomials such as the following: Suppose that $f(x)$ is a polynomial with integral coefficients and that the least upper bound of its zeros is not positive. Take $t=10^m$ where m is the number of digits of the greatest coefficient of $f(x)$. Then it is necessary and sufficient for the reducibility of $f(x)$ that $f(t)$ should have at least two factors (each greater than unity) and that the sum of the coefficients of the number $f(t)$ when expanded in the scale of t should equal the product of the corresponding sums for the two factors. A generalisation of the theorem to products of r polynomials is also given.

R. A. Rankin (Birmingham).

Yakovkin, M. V. On a method of finding irreducible factors. Doklady Akad. Nauk SSSR (N.S.) **93**, 783–785 (1953). (Russian)

The criterion of the paper reviewed above is applied to the polynomial

$$x^5-5x^4+13x^3-22x^2+27x-20.$$

This leads to the consideration of the number

$$10513222720=1020305\cdot 10304.$$

This factorisation has the required property and gives rise to the factors x^3-2x^2+3x-5 and x^2-3x+4. This example is one to which Chebotarev applied Kronecker's method of reduction using a great number of steps.

R. A. Rankin.

Chamberlin, E., and Wolfe, J. Note on a converse of Lucas's theorem. Proc. Amer. Math. Soc. **5**, 203–205 (1954).

Referring to the convex closure of the zeros of a polynomial $p(z)$ of degree n as its Lucas polygon π, then the Lucas Theorem states that $\pi'\subset\pi$, where π' is the Lucas polygon for $p'(z)$, the derivative of $p(z)$. The symbol π_0 is used to denote the intersection of the Lucas polygons $\pi(c)$ of $p(z)+c$ for all constants c. Then $\pi'\subset\pi_0$ also. The authors prove that, if a side Σ of $\pi(c)$ is determined by just two simple zeros of $p(z)+c$, then $\Sigma\cdot\pi_0\neq 0$ unless $n=2$. They also show by example some of the difficulties involved when similar information is sought concerning a side Σ that contains multiple zeros or additional simple zeros. Reference

is given to J. L. Walsh, "The location of critical points of analytic and harmonic functions" [Amer. Math. Soc. Colloq. Publ., v. 34, New York, 1950, pp. 71–72; these Rev. 12, 249]. *M. Marden* (Milwaukee, Wis.).

Turán, P. Hermite-expansion and strips for zeros of polynomials. Arch. Math. 5, 148–152 (1954).

As counterparts of two well known results due to Cauchy and Walsh, respectively, the author obtains the following upper bounds for the imaginary part y of the zeros of the polynomial

$$e^{z^2}\sum_{m=0}^{n} b_m(e^{-z^2})^{(m)}.$$

Let $b_n\neq 0$, max $|b_m|=M$, $m=0, 1, \cdots, n-1$; then

$$2|y|\leq 1+|b_n|^{-1}M,\quad 2|y|\leq \sum_{\nu=1}^{n}|b_n^{-1}b_{n-\nu}|^{1/\nu}.$$

G. Szegö (Stanford, Calif.).

Parodi, Maurice. Sur les polynomes d'Hurwitz. C. R. Acad. Sci. Paris 238, 1466–1467 (1954).

Let $f(x)=a_0x^n+\cdots$, $a_0>0$, be a real polynomial of degree n with zeros in the left half-plane ("Hurwitz polynomial"), $g(x)=b_0x^n+\cdots$, $b_0>0$, real and of degree n. The purpose of the author is to find $d=\sup|\lambda|$ where λ is real and such that $f(x)+\lambda g(x)$ is a Hurwitz polynomial. As is known, $f(x)$ is a Hurwitz polynomial if and only if a certain matrix $M(f)$ has positive principal minors $M_k(f)$. Let δ_k^{-1} be the sum of the absolute values of the elements of $[M_k(f)]^{-1}$ and $\beta_k=\max|b_k|$ where b_k are all coefficients of $g(x)$ occurring in $M_k(g)$. The author announces the result that

$$d=\min[b_0^{-1}a_0, \beta_k^{-1}\delta_k],\quad k=1, 2, \cdots, n.$$

G. Szegö (Stanford, Calif.).

Special Functions

Eberlein, W. F. The elementary transcendental functions. Amer. Math. Monthly 61, 386–392 (1954).

Fempl, Stanimir. Über einige Reduktionen des vollständigen elliptischen Normalintegrals III Gattung. Srpska Akad. Nauka. Zbornik Radova 35. Mat. Inst. 3, 129–146 (1953). (Serbo-Croatian. German summary)

It is well known that the complete elliptic integral of the third kind

$$\text{(1)}\qquad \int_0^{\pi/2}(1+\nu\sin^2\phi)^{-1}(1-k^2\sin^2\phi)^{-1/2}d\phi$$

can be reduced to incomplete elliptic integrals of the first and second kinds (Legendre). The author shows that (1) can be reduced to complete elliptic integrals of the first and second kinds (and hence is much easier to compute) if the modulus k and the parameter ν satisfy certain algebraic relations. He gives explicitly six cases of such reductions. *A. Erdélyi* (Pasadena, Calif.).

Cazenave, René. Intégrales et fonctions elliptiques usuelles. Ann. Télécommun. 9, 103–108 (1954).

After a brief review of the historical origin of elliptic functions, and of the principal formulas relating to them, the author indicates the application of elliptic functions to the parametrization of quartic plane curves, and also the connection of elliptic functions with spherical trigonometry. *A. Erdélyi* (Pasadena, Calif.).

Thosar, Yeshwant V. Generalisations of Neumann's formula for $Q_n(y)$. Math. Z. 60, 52–60 (1954).

The author "generalizes" Neumann's formula

$$Q_n(y)=\frac{1}{2}\int_{-1}^{1}\frac{P_n(x)}{y-x}dx$$

by (i) differentiating it r times, (ii) integrating by parts m times, and (iii) combining the two operations. The integrals thus obtained are used to evaluate some further integrals, and to sum some infinite series involving products or squares of Legendre functions of the second kind. *A. Erdélyi.*

Bagchi, Hari das, and Mukherjee, Bhola nath. Note on the operational representations of some special functions. Math. Z. 60, 88–93 (1954).

The authors use operational calculus to derive a considerable number of formulas for Hermite polynomials and for Palamà's second solution of Hermite's equation [Boll. Un. Mat. Ital. (3) 5, 72–77 (1950); these Rev. 12, 25]. The authors state that "the paper is believed to embody some amount of original matter"; the reviewer was unable to identify anything that would substantiate the authors' belief. *A. Erdélyi* (Pasadena, Calif.).

Poli, L. Fonctions hypergéométriques et calcul symbolique. Ann. Univ. Lyon. Sect. A. (3) 16, 37–51 (1953).

The author remarks that in many linear relations involving generalized hypergeometric series, parameters which occur in all the series involved may be dropped, or added, without destroying the truth of the relation. The process may be justified by applying the Laplace transformation, or the inverse Laplace transformation. The author gives many examples of this process. A typical example is as follows:

$$k^a\cdot{}_1F_0(a;kx)=\sum_0^\infty\binom{a+n-1}{n}(1-k^{-1})^n{}_1F_0(a+n;x)$$

is a true relation (it being a disguised form of the binomial theorem). Adding two parameters, this relation becomes

$$k^a\cdot{}_2F_1(a,b;c;kx)=\sum_0^\infty\binom{a+n-1}{n}(1-k^{-1})^n{}_2F_1(a+n,b;c;x),$$

a relation which "seems, at first sight, very complicated." *A. Erdélyi* (Pasadena, Calif.).

Meijer, C. S. Expansion theorems for the G-function. VI. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 77–82 (1954).

Meijer, C. S. Expansion theorems for the G-function. VII. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 83–91 (1954).

[For parts I to V see these Rev. 14, 469, 642, 748, 979; 15, 422.] Part VI contains a number of lemmas which are used in Part VII to prove the fourth and fifth expansion theorems. These are of a form somewhat similar to the first two expansion theorems except that ${}_{h+1}F_{h+1}$ appears instead of ${}_{h+1}F_h$. *A. Erdélyi* (Pasadena, Calif.).

Rathie C. B. Some infinite integrals involving E-functions. J. Indian Math. Soc. (N.S.) 17 (1953), 167–175 (1954).

The author evaluates numerous integrals whose integrands contain Bessel functions, confluent hypergeometric

functions, generalized hypergeometric series, MacRobert's E-function. The results are too numerous and too involved to be particularized here. *A. Erdélyi.*

Toscano, Letterio. Le funzioni del cilindro parabolico come caso limite delle funzioni ipergeometriche. Boll. Un. Mat. Ital. (3) 9, 29–38 (1954).

The author shows that

$$\lim_{s\to\infty} \frac{\Gamma(s+\nu)}{s^{\nu/2}\Gamma(s)} {}_2F_1\left(-\nu, s+k+\nu; \frac{s+k+1}{2}; \frac{1-s^{-1/2}x}{2}\right) = \exp\left(\tfrac{1}{4}x^2\right)D_\nu(x)$$

and gives several particular cases and applications of this result. *A. Erdélyi* (Pasadena, Calif.).

Nagase, Masahumi. Asymptotic expansions of Bessel functions in the transitional regions. J. Phys. Soc. Japan 9, 296–297 (1954).

Let $\nu = z\cosh\gamma$, $\xi = i\nu/3\tanh^3\gamma$, $\eta = \tanh\gamma$ (z and ν complex). The author refines the asymptotic formulas for Bessel functions $H_\nu^{(j)}(z)$ of the third kind to terms of order η^8. To do this assumed expansions of the form

$$H_\nu^{(j)}(z) = 3^{-1/2}e^{\pm(\pi/3)i}\left\{\eta H_{-1/3}^{(j)}(\xi) + \sum_{n=1}^{\infty} a_n^{(j)}(\xi)\eta^{2n+1}\right\}$$

are substituted into Bessel's equation and $a_n^{(j)}(\xi)$ computed for $n \leq 4$. *N. D. Kazarinoff* (Lafayette, Ind.).

Cooke, J. C. Note on some integrals of Bessel functions with respect to their order. Monatsh. Math. 58, 1–4 (1954).

The several integrals investigated are evaluated using Fourier integrals. For example, from the definition of Anger's function $\mathbf{J}_\xi$ a Fourier cosine integral for $\int_{-\infty}^{+\infty}\mathbf{J}_\xi(z)\cos(a\xi)d\xi$ can be obtained directly. When $a=0$, this yields the simple result $\int_{-\infty}^{+\infty}\mathbf{J}_\xi(z)d\xi = 1$. The misprint $1/z$ for $1/2$ appears twice. *N. D. Kazarinoff* (Lafayette, Ind.).

Harmonic Functions, Potential Theory

Garabedian, P. R. Applications of analytic continuation to the solution of boundary value problems. J. Rational Mech. Anal. 3, 383–393 (1954).

L'auteur développe diverses méthodes particulières réalisant la réflexion des solutions des équations $\Delta u = u$, $\Delta\Delta u = 0$, $\Delta\Delta u = 2\Delta\partial u/\partial x$, $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, dans un arc analytique de la frontière du domaine où elles sont définies. Les résultats obtenus permettent, dans certains cas, d'obtenir la solution explicite de problèmes aux limites et d'écrire la fonction de Green correspondante. A titre d'exemples, l'auteur indique comment résoudre le problème de Dirichlet pour $\Delta u = u$ dans un angle et comment déterminer la fonction de Green de l'équation $\Delta\Delta u = 0$ pour l'extérieur d'une ellipse ou le croissant formé par deux arcs de cercle, les conditions à la frontière étant $u = \partial u/\partial n = 0$. Il signale plusieurs autres problèmes aux limites que ses méthodes permettent d'aborder. *H. G. Garnir* (Liège).

Dinghas, Alexandre. Sur quelques inégalités concernant une classe d'intégrales de Dirichlet. C. R. Acad. Sci. Paris 237, 639–641 (1953).

Let S_n denote the sphere $\xi_1^2 + \cdots + \xi_n^2 = 1$, and let C^n be a manifold for which $ds^2 = dl^2 + \psi^2(l)(d\xi_1^2 + \cdots + d\xi_n^2)$, where K is a positive constant, $0 \leq K^{\frac{1}{2}} \leq \pi/l$, and where $\psi(l) \equiv K^{-\frac{1}{2}}\sin K^{\frac{1}{2}}l$. If $P(l, \xi_1, \cdots, \xi_n) \in C^n$, if $d\sigma_n$ is the volume element of C^n, and if $v(P)$ is a smooth non-negative function defined on a set $A \subset C^n$, and vanishing on the boundary of A, then the integral $J(v) \equiv 2\int_A v\,d\sigma_n$ can be interpreted as a volume integral $V(M)$ of a figure M in the space Z^{n+1} for which $ds_1^2 = ds^2 + dy^2$, $-\infty < y < \infty$, and which is symmetric about $y=0$. Set $\tau(l) \equiv \omega_n\int_0^l \psi^{n-1}(\alpha)d\alpha$, and set $\sigma(l) \equiv \tau(l)/\tau'(l)$. Then under certain conditions of regularity of A, the author uses the expression

$$\sigma(M) = 2\int_A [1+\nabla(v)]^{\frac{1}{2}}d\sigma_n,$$

where ∇ is the first Beltrami operator in C^n, to obtain

$$(1)\quad q\int_A [1+\nabla(v)]^{\frac{1}{2}}d\sigma_n \geq \int_A v\,d\sigma_n + 2\omega_n\int_0^{l_0}[q^2 - \sigma^2(\alpha)]^{\frac{1}{2}}\psi^{n-1}(\alpha)d\alpha,$$

where $q \equiv \sigma(l_0)$ and where l_0 is defined by $\int_A d\sigma_n = \tau(l_0)$. The author then obtains

$$(2)\quad q\int_A \nabla(v)^{\frac{1}{2}}d\sigma_n \geq \int_A v\,d\sigma_n.$$

The author states that (1), (2) can be used to estimate the order of harmonic and subharmonic functions in E^n. *M. O. Reade* (Ann Arbor, Mich.).

Šerman, D. I. On a singular problem from potential theory. Doklady Akad. Nauk SSSR (N.S.) 94, 25–28 (1954). (Russian)

Let S be a bounded simply connected domain in the plane of $z = x+iy$, whose frontier L is a suitably smooth closed contour L; the origin is taken in S. One is to find u, v, harmonic in S, continuous with their first partials on $S+L$, such that on L

$$(1)\quad u_s - v_\nu + a_{11}u + a_{12}v = f_1,\quad u_\nu + u_s + a_{21}u + a_{22}v = f_2,$$

where the coefficients are assigned, suitably differentiable functions of arc s; $0 = a_{11} + a_{22} = a_{21} - a_{12}$ at no point of L. Here u and v are sought in terms of certain potentials, whose real densities satisfy a Fredholm system. The author leaves open the question as to whether two harmonic functions are expressible in the form considered, as well as the question as to the conditions under which problem (1) can be solved. *W. J. Trjitzinsky* (Urbana, Ill.).

van der Vaart, H. R. An elementary method of expressing the Laplacian Δc in terms of curvilinear non-orthogonal coordinates, with some corollaries. Simon Stevin 30, 48–57 (1954).

Hill, E. L. The theory of vector spherical harmonics. Amer. J. Phys. 22, 211–214 (1954).

Let r, θ, ϕ be spherical polar coordinates, $\mathbf{r}_1$, $\boldsymbol{\theta}_1$, $\boldsymbol{\phi}_1$ unit vectors in the direction of increasing r, θ, ϕ, and let

$$Y_l^m(\theta,\phi) = (-1)^m\left[\frac{2+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi}$$

be the normalized (scalar) spherical surface harmonics.

Three types of vector surface harmonics are defined in this paper:

$$\mathbf{V}_{lm}=[(l+1)(2l+1)]^{-1/2}\times\left[-(l+1)Y_l^m\mathbf{r}_1+\frac{\partial Y_l^m}{\partial\theta}\boldsymbol{\theta}_1+im\operatorname{cosec}\theta Y_l^m\boldsymbol{\phi}_1\right]$$

$$\mathbf{X}_{lm}=[l(l+1)]^{-1/2}\left[-m\operatorname{cosec}\theta Y_l^m\boldsymbol{\theta}_1-i\frac{\partial Y_l^m}{\partial\theta}\boldsymbol{\phi}_1\right]$$

$$\mathbf{W}_{lm}=[l(2l+1)]^{-1/2}\left[lY_l^m\mathbf{r}_1+\frac{\partial Y_l^m}{\partial\theta}\boldsymbol{\theta}_1+im\operatorname{cosec}\theta Y_l^m\boldsymbol{\phi}_1\right].$$

These form an orthonormal system. *A. Erdélyi.*

Differential Equations

Have

★**Tricomi, Francesco Giacomo. Equazioni differenziali.** 2a ed. Edizioni Scientifiche Einaudi, Torino, 1953. 353 pp. 4000 Lire.

[For a review of the first edition of this book, see these Rev. 10, 375.] In this second edition the general plan of the book has been left unchanged, but some additional material has been inserted, and the details of the exposition have been revised in many places. Among the more important changes are the following. (1) The presentation of the Poincaré-Bendixson theory of the qualitative properties of the solutions of a first-order equation has been expanded somewhat, and the applications of that theory to the theory of non-linear vibrations are discussed briefly. (2) Numerous improvements have been made in the chapter dealing with the asymptotic properties of the solutions of linear equations. These include a simpler and deeper treatment of the solutions of the equation $y''+Q(x)y=0$, and a revised discussion of the determination of eigenvalues in the case of Legendre's equation. (3) The bibliography has been amplified, and some material which was treated in an appendix in the first edition has been transferred to the main text. All of these changes have been wisely made, and they enhance the value of this already excellent text.

L. A. MacColl (New York, N. Y.).

Lozinskiĭ, S. M. On the interval of existence of a solution of a system of ordinary differential equations. Doklady Akad. Nauk SSSR (N.S.) **94**, 17–19 (1954). (Russian)

This is a continuation of previous papers [same Doklady (N.S.) **92**, 225–228 (1953); **93**, 621–624 (1953); these Rev. **15**, 473, 651] with a parallel set of four theorems relating to conditions under which the solution will lie in a specified region. No proofs are given. *A. S. Householder.*

Haas, Felix. The global behavior of differential equations on n-dimensional manifolds. Proc. Nat. Acad. Sci. U. S. A. **39**, 1258–1260 (1953).

Let M_n be a compact, connected, $C^{(2)}$ n-manifold and V a tangent vector field on M_n satisfying a Lipschitz condition. Let C be an integral curve of V with positive semi-characteristic C^+ and ω-limit set $\bar{C}$ which contains no singular points of V. The author announces the following theorems (notation that of the reviewer). Theorem: If $\bar{C}$ is somewhere dense in M_n, then $\bar{C}=M_n$, M_n has Euler characteristic $\chi(M_n)=0$, and V has no singular points. Theorem: Let $M_n=M_{n-1}\times S^1$ and $\chi(M_n)\neq 0$; if the component of V along S^1, $d\theta/dt$, is nowhere zero on M_n, then there is a periodic integral curve of V. The sketch of the proof offered for the first of these theorems is too brief to convince the reviewer.

L. Markus (New Haven, Conn.).

Haas, Felix. Poincaré-Bendixson type theorems for two-dimensional manifolds different from the torus. Ann. of Math. (2) **59**, 292–299 (1954).

Let V be a $C^{(2)}$ differentiable tangent vector field on a differentiable, compact, orientable 2-manifold M. Let C be an integral curve of V with positive semi-characteristic C^+ and ω-limit set $\bar{C}$ which contains no singular (critical) points of V. If M is the torus and V is non-singular, Siegel [Ann. of Math. (2) **46**, 423–428 (1945); these Rev. **7**, 117] discussed and extended the classical results of Denjoy. For all other cases the author shows that $\bar{C}$ is a closed characteristic and that either $C^+=\bar{C}$ or C^+ spirals towards $\bar{C}$ from one side. For certain statements necessary in the proof of the principal Theorem 4, i.e., the map φ of B into S is sense-preserving, and $\bar{C}$ is nowhere dense in M, the author refers to an earlier work of his [Proc. Amer. Math. Soc. **4**, 630–636 (1953); these Rev. **15**, 126]. *L. Markus.*

Ehlers, Georg. Über schwach singuläre Stellen linearer Differentialgleichungssysteme. Arch. Math. **3**, 266–275 (1952).

The author uses a compact notation to express a fundamental solution matrix $Y=(\sum_{\nu=0}^{\infty}C_\nu t^\nu)t^B$ of the singular differential system

$$t\frac{dY}{dt}=\left(\sum_{\nu=0}^{\infty}A_\nu t^\nu\right)Y=A(t)Y.$$

Here the C_ν, A_ν, and B are $n\times n$ complex matrices, and the power series have the same radii of convergence.

L. Markus (New Haven, Conn.).

Babkin, B. N. Solution of a boundary problem for an ordinary differential equation of second order by Čaplygin's method. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 239–242 (1954). (Russian)

Consider $y''-f(x,y)=0$, $y(a)-A=y(b)-B=0$, with $f_y(x,y)\geqq 0$ for $a\leqq x\leqq b$ and all y. For any $z(x)$ of class C'' satisfying $z(a)-A=z(b)-B=0$, let $\alpha(x)=z''-f(x,z)$. Then $v=z+\eta$ is an upper function and $u=z+\tau$ is a lower function if η satisfies $\eta''+|\alpha(x)|=0=\eta(a)=\eta(b)$ and τ satisfies $\tau-|\alpha(x)|=0=\tau(a)=\tau(b)$. Setting $v=v_0$ and $u=u_0$, the author now forms in sequence $v_{n+1}=v_n-\delta_n$, where

$$\delta_n''-M\delta_n-\alpha_n=0=\delta_n(a)=\delta_n(b),\quad \alpha_n=v_n''-f(x,v_n),$$

and $M=\sup f_y$ in the region bounded by the curves $y=u(x)$ and $y=v(x)$. This and the sequence u_n formed analogously are shown to be sequences of upper and lower functions, respectively, which converge uniformly and monotonically to the solution y. *A. S. Householder.*

Lewis, D. C. Simple operational equations with constant coefficients. Math. Mag. **27**, 177–188 (1954).

Let C be a linear space of functions $x(t)$, t in some set S, and let L be a linear operator which maps C into itself. Suppose there is a function $\phi_\lambda(t)\in C$ which is an analytic function of λ in some region R of the complex plane. Suppose also that $\phi_\lambda(t)$ satisfies the conditions: (1) $d^p\phi_\lambda(t)/d\lambda^p\in C$, $p=1,2,\cdots$; (2) $\phi_\lambda(t)\neq 0$ for $t\in S$ and $\lambda\in R$;

$$(3)\qquad g_{p\lambda}=[\phi_\lambda(t)]^{-1}d^p\phi_\lambda(t)/d\lambda^p,$$

$p=0,1,2,\cdots$ are linearly independent for each fixed λ; (4) $L\phi_\lambda(t)=\lambda\phi_\lambda(t)$; (5) $Ld^p\phi_\lambda(t)/d\lambda^p=d^pL\phi_\lambda(t)/d\lambda^p$. Con-

sider the system of operational equations

$$(*)\qquad \sum_{j=1}^{m} f_{ij}(L)x_j(t)=0,\quad i=1, 2, \cdots, m,$$

where the $f_{ij}(L)$ are polynomials in L with constant coefficients. Suppose $\Delta(\lambda)=\det f(\lambda)$ where the matrix $f(\lambda)=(f_{ij}(\lambda))$ is of degree greater than zero and has $\lambda_1 \varepsilon R$ as an s-fold zero. Then each column of the matrices

$$(d^p[\phi_\lambda(t)F(\lambda)]/d\lambda^p)_{\lambda=\lambda_1},\quad p=0, 1, \cdots, s-1,$$

where $F(\lambda)=\operatorname{adj} f(\lambda)$ (transposed matrix of cofactors) is a solution of (*) and there are exactly s linearly independent such columns. The author's object is "to give a unified treatment of systems of homogeneous linear differential and difference equations with constant coefficients", but he points out that "all the results must be considered as essentially known". The algebraic difficulties are treated by showing that the square matrix of order ms formed from matrices of order m, the block in the ith row and jth column being 0 if $i>j$ and equal to $\binom{j-1}{i-1}F^{(j-i)}(\lambda_1)$ if $i\leqq j$, $i, j=1, 2, \cdots, s$, is of rank s. *C. E. Langenhop.*

Persidskiĭ, K. On characteristic numbers. Izvestiya Akad. Nauk Kazah. SSR **1952**, no. 116, Ser. Astr. Fiz. Mat. Meh. 1(6), 64–76 (1952). (Russian. Kazak summary)

The title of the paper is somewhat misleading. It deals mainly with a differential equation

$$(1)\qquad \dot{x}=L(t;x)+N(t;x)$$

where x is an element of a complete normed linear space, L is a linear operator relative to x and $M=o(\|x\|)$. First the theory is developed when $N=0$ (linear homogeneous case) or $N=f(t)$ (linear non-homogeneous case). The general case is then attacked and it is shown that the stability at the origin is governed by that of the first approximation

$$(2)\qquad \dot{x}=L(t;x)$$

if and only if every solution $x(t)$ of the latter satisfies an inequality for $t\geqq t_0>0$: $\|x(t)\|\leqq B\|x(t_0)\| \exp\{-x(t-t_0)\}$ where x, B are positive constants.

The Lyapunov characteristic numbers of the solutions of (2) are discussed under the assumption that $\|L(t;x)\|\leqq\lambda\|x\|$. Their set or "spectrum" is shown to belong to the interval $[-\lambda, \lambda]$. Theorem: If the spectrum has no points in $[-\infty, r]$ then every solution $x(t)$ satisfies

$$\|x(t)\|\leqq\|x(0)\|B(r)\exp(-rt).$$

Notice that if $r>0$ then the origin is asymptotically stable for the system (1).

Various theorems proved for the case of a space with countable components and whose proofs are unchanged are recalled by the author. [Relevant references: Persidskiĭ, Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 229–240 (1949); **14**, 635–650 (1950); Izvestiya Akad. Nauk Kazah. SSR **1950**, no. 97, Ser. Mat. Meh. 4, 3–18 (1950); these Rev. **11**, 33; **12**, 500; **14**, 753.] *S. Lefschetz* (Princeton, N. J.).

Hayl

★**Bellman, Richard. Stability theory of differential equations.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. xiii+166 pp. $5.50.

The scope of this book is somewhat wider than its title may suggest. It is concerned with the behavior of solutions of real differential equations as the independent variable t tends to infinity. Chapter 1 begins with a brief account of matrix theory, followed by the theory of the linear system (1) $dx/dt=A(t)x$: existence of solutions, solution of the inhomogeneous equation by variation of constants, a proof that all solutions of $dx/dt=Ax$ tend to zero as $t\to\infty$ if and only if the characteristic roots of A have negative real parts, the general form of solution for (1) when $A(t)$ is periodic. The Jordan canonical form of a matrix is described, but not used; the triangular form is used if possible, and otherwise distinct characteristic roots are assumed if multiple roots cause undue complication. Chapter 2 gives the reader a first taste of stability theory. The main problem treated is: do solutions of (2) $dx/dt=(A+B(t))x$, where $B(t)$ is small when $t\to\infty$, behave like those of $dx/dt=Ax$ when $t\to\infty$? A useful technique, much used later in the book, is introduced here: derive an integral equation

$$x(t)=e^{At}x(0)+\int_0^t e^{A(t-s)}B(s)x(s)ds$$

for solutions of (2), and use this to compare $x(t)$ with $e^{At}x(0)$ for large t. Chapter 3 presents standard local existence theorems for $dx/dt=f(x, t)$, both when f satisfies a Lipschitz condition, and when it is merely continuous. However, no reference is made to the fact that these local solutions may be pieced together to global solutions. Chapter 4 deals mainly with the equation $dx/dt=Ax+f(x)$, where $\|f(x)\|=O(\|x\|)$ as $x\to 0$. The "fundamental stability theorem" proved here is: if the characteristic roots of A have negative real parts, then all solutions with sufficiently small initial value $x(0)$ tend to zero as $t\to\infty$. There follow theorems which say what happens when some roots have positive real parts, and also when the constant matrix A is replaced by a variable matrix $A(t)$. The author has succeeded here in giving a simple account of a subject which is often made to look alarmingly difficult. Chapter 5 describes some remarkable results of Hardy (which followed earlier work by Borel and Lindelöf) on the rate growth as $t\to\infty$ of solutions of a polynomial equation $P(t, x, dx/dt)=0$. For instance, if R is a rational function of x and t, any solution of $dx/dt=R(x, t)$ which exists for all large t is ultimately monotonic and so are all its derivatives; also, either $x\sim At^a e^{P(t)}$, where P is a polynomial, or $x\sim At^a(\log t)^{1/n}$, where n is an integer. This is a valuable chapter, not only because it contains interesting results which are probably little known, but also because it shows how much information about solutions of a differential equation can be obtained directly from the equation, without using explicit analytical expressions for the solutions, a point seldom brought out by standard textbooks. The same lesson is driven home by Chapter 7, which treats the Emden-Fowler equation $d^2x/dt^2\pm tx=0$ and gives a fairly precise description of the possible asymptotic behaviour of solutions as $t\to\infty$. In Chapters 5 and 7, some cooperation by the reader is called for in supplying details which are only sketched by the author. Chapter 6 deals with second-order linear equations, usually in the normal form $d^2x/dt^2+p(t)x=0$. After discussing "qualitative" questions (e.g., are the solutions bounded, are they in $L^2(0, \infty)$, have they infinitely many zeros?) the author turns to the equations $d^2x/dt^2\pm(1+\phi(t))x=0$ with $\phi(t)$ small for large t. Particular attention is paid to those asymptotic formulae for the solutions which cannot be derived from the general results on systems proved in Ch. 2. A specimen result is: if $\phi(t)\to 0$ as $t\to\infty$ and $\int_0^\infty\phi^2(t)dt<\infty$, then $d^2x/dt^2-(1+\phi(t))x=0$ has two solutions whose asymptotic forms are $x\sim\exp[\pm(t+\int_0^t\phi(s)ds)]$. The author's stated aim in this chapter is "to present a cross-section of theorems and techniques", and he has succeeded both in giving a fair sample of what is known and

in showing the more useful tricks of the trade to the reader. A similar spirit pervades the whole book: space is used to prove theorems several times if this exhibits several useful methods, but never to give an exhaustive catalogue of known results. There are plenty of exercises and references which the reader may use to extend his knowledge; perhaps some exercises might have been accompanied by hints for solution or detailed references, since they seem too difficult for unaided solution by non-expert readers. The book is to be judged not as a monograph but as an advanced textbook: as such, it is excellent both in its choice of material (most of which is not to be found in standard textbooks on differential equations) and in its lucid and attractive manner of presentation. *G. E. H. Reuter* (Manchester).

Krasovskiĭ, N. N. On stability of solutions of a system of second order in critical cases. Doklady Akad. Nauk SSSR (N.S.) **93**, 965–967 (1953). (Russian)

A stability criterion is established for a borderline case for the equations $\dot{x}_i = X_i(x_1, x_2)$ $(i=1, 2)$. It is assumed that the X_i vanish at $(0, 0)$ and have partial derivatives satisfying a Lipschitz condition in a neighborhood of $(0, 0)$. Let $p(\lambda; x_1, x_2) = \det(\|\partial X_i/\partial x_j\| - \lambda I)$, where I is the unit matrix and let $p(\lambda; 0, 0)$ have two imaginary roots or one zero root and one negative root. It is then proved that if, for each (x_1, x_2) other than $(0, 0)$, $p(\lambda; x_1, x_2)$ has roots with negative real parts, then $(0, 0)$ is asymptotically stable, whereas, if, at each (x_1, x_2), $p(\lambda; x_1, x_2)$ has roots with positive real parts, then $(0, 0)$ is unstable. *W. Kaplan.*

Kamenkov, G. V. On stability of motion over a finite interval of time. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 529–540 (1953). (Russian)

Let $\dot{x} = X(x, t)$, $X(0, t) = 0$, be a system of differential equations. The author considers the following rather peculiar problem: Is it possible to find a quadratic positive definite form $V(x)$ in the components of x and two positive numbers A, τ such that from $V[x(t_0)] \leqq A$ follows $V[x(t)] \leqq A$ for all $t_0 \leqq t \leqq t_0 + \tau$? If this is so the system is said to be "stable over a finite interval of time". If $L(t)$ is the matrix of the linear part of X, the results obviously to be expected are: if the characteristic roots of $L(t_0)$ have negative real parts, the system is "stable" in the above mentioned sense; if some roots have positive real parts, the system is "unstable"; if the real parts are $\leqq 0$, some of them being $=0$, the system may be "stable" or not depending on the higher order terms. The reviewer makes the two following critical remarks. (1) The long and complicated proofs given in this paper may be reduced to a few lines by using the so-called "second method of Lyapunov" and certain well-known algebraic lemmas [cf. I. G. Malkin, Theory of stability of movement, Gostehizdat, Moscow-Leningrad, 1952, p. 62, Theorem 1]. (2) The author assumes (Theorem 4) that if multiple characteristic roots exist some complementary conditions are needed to insure "stability", which is false.

J. L. Massera (Montevideo).

Šimanov, S. N. On the stability of the solution of a non-linear system of equations. Uspehi Matem. Nauk (N.S.) **8**, no. 6(58), 155–157 (1953). (Russian)

Consider the equation $\ddot{x} + f(x, \dot{x})\dot{x} + \varphi(x) = 0$, where f and φ are continuous and satisfy the uniqueness condition for all real values of the arguments. Now the solutions of $\ddot{x} + h\dot{x} + kx = 0$ are asymptotically stable provided that the constants h, k are positive. Supposing $f > 0$, $\varphi/x > 0$, what further conditions are to be imposed to guarantee asymptotic stability of the origin in the phase plane? Here the case of two equations of the second order is dealt with and the following result obtained. Given the system

$$\ddot{x} + f_1(x, y, \dot{x}, \dot{y})\dot{x} + \varphi(x) = 0,$$
$$\ddot{y} + f_2(x, y, \dot{x}, \dot{y})\dot{y} + \psi(y) = 0.$$

Suppose that: $\varphi(0) = \psi(0) = 0$; $\varphi(x)/x$, $\psi(y)/y > 0$ for $x, y \neq 0$; $f_1 > 0$, $f_2 > 0$; $\int_0^x \varphi(x)dx$, $\int_0^y \psi(y)dy \to \infty$ with $|x|$, $|y|$. Then the origin (in the phase space) is asymptotically stable whatever the initial position. There is an obvious generalization. *S. Lefschetz* (Princeton, N. J.).

Miller, Kenneth S. A remark on stability. J. Appl. Phys. **25**, 407–408 (1954).

The stability property in question is that of the absolute integrability over $(0, \infty)$ of all solutions of an nth order linear homogeneous differential equation $Lu = 0$. It is shown, under restrictions, that this is equivalent to the property $\int_0^\infty |H(t, y)| dt < \infty$, for all y in $[0, \infty)$, where $H(t, y)$ is the "one-sided Green's function" of L [cf. the author, same J. **22**, 1054–1057 (1951); these Rev. **13**, 348]. The latter condition cannot be replaced by

$$\int_0^\infty |H(t, y)| dy < \infty.$$

F. V. Atkinson (Ibadan).

Popovici, C. Stabilité pondérée. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 243–261 (1952). (Romanian. Russian and French summaries)

The author discusses the system of ordinary differential equations $dx/dt = v(x)$ where x is an n-dimensional vector and $v(x)$ is a vector function of x. A singular point of the vector-field v, i.e., a point x_0 at which v vanishes, is a position of equilibrium of the system. The equilibrium is in general unstable, but it may happen that in the immediate neighborhood of the point of equilibrium there is an invariant subset of the differential system, i.e., a region such that any trajectory through a point of the region is entirely in the region. In this case the author speaks of weighted stability. In such an invariant subregion there must be points of stable equilibrium, and the author develops methods for discovering the number and character of these. He also discusses the stability of the trajectories. *A. Erdélyi.*

Levin, J. J., and Levinson, Norman. Singular perturbations of non-linear systems of differential equations and an associated boundary layer equation. J. Rational Mech. Anal. **3**, 247–270 (1954).

The solutions of a "full" system

$$\text{(F)} \qquad \dot{x} = f(x, y, t, \epsilon), \quad \epsilon\dot{y} = g(x, y, t, \epsilon),$$

are compared, for small $\epsilon > 0$, with those of the "degenerate" system (D): $\dot{x} = f(x, y, t, 0)$, $0 = g(x, y, t, 0)$. Here x, y are vectors of dimensions n_0, n, and $r > 0$ is constant. (D) is assumed to have a solution $x = \phi(t)$, $y = \psi(t)$, of class C' in $a \leqq t \leqq b$; f and g are assumed of class C' in all arguments in a set R: $|x - \phi(t)| < \delta_0$, $|y - \psi(t)| < \delta_0$, $a \leqq t \leqq b$, $0 \leqq \epsilon < \delta_0$. An essential further hypothesis is (H): the characteristic roots of the matrix $\partial g_i/\partial y_j(\phi(t), \psi(t), t, 0)$ have real parts $\leqq -\mu < 0$ when $a \leqq t \leqq b$. Then (i) if $\epsilon + |\xi - \phi(a)| + |\eta - \psi(a)|$ is small enough, (F) has a unique solution with initial values $x(a) = \xi$, $y(a) = \eta$; this solution exists on $a \leqq t \leqq b$, and tends uniformly to the degenerate solution $x = \phi(t)$, $y = \psi(t)$ as $\epsilon = |\xi - \phi(a)| + |\eta - \psi(a)| \to 0$. (ii) There exists $\delta > 0$, independent of ϵ, such that if $|\eta - \psi(a)| < \delta$ and ξ is small enough, (F) has a unique solution on $a \leqq t \leqq b$ with initial

values $x(a)=\phi(a), y(a)=\eta$; as $\epsilon\to 0$, this solution tends to the degenerate solution uniformly on any interval $c\leqq t\leqq b$, $c>a$. If (H) is slightly strengthened, (ii) can be improved: the full solution has its y, near $t=a$, very close to the Y obtained from the "boundary-layer equation" $\epsilon^r\dot{Y}=g(\phi(a), Y, a, 0)$ with $Y(a)=\eta$, and this fact enables the δ appearing in (ii) to be determined fairly explicitly. The above results are finally generalized to the system $\dot{x}=f(x, y, z, t, \epsilon)$, $\epsilon^r\dot{y}=g(x, y, z, t, \epsilon)$, $\epsilon^s\dot{z}=h(x, y, z, t, \epsilon)$ and its associated degenerate system. The authors compare their hypotheses with those of A. Tihonov [Mat. Sbornik N.S. **27(69)**, 147–156 (1950); these Rev. **12**, 181] and find that neither set of hypotheses implies the other. A note added in proof refers to similar results by I. S. Gradšteĭn [Doklady Akad. Nauk SSSR (N.S.) **64**, 441–443; **65**, 789–792; **66**, 789–792 (1949); these Rev. **10**, 536, 708, 709]. *G. E. H. Reuter.*

Taam, Choy-Tak. The boundedness of the solutions of a nonlinear differential equation. Proc. Amer. Math. Soc. **5**, 122–125 (1954).

The main result may be re-stated thus: Every solution of $(r(x)y')'+p(x)y=f(x, y)$ exists and is bounded on $0\leqq x<\infty$, provided that (i) $1/r(x)$, $p(x)$ and $f(x, y)$ (for fixed y) are real-valued and in $L(0, R)$ for each $R>0$; (ii) rp is positive and AC; (iii) $(rp)^{-1}\min[(rp)', 0]$ is in $L(0, \infty)$; (iv) $|f(x, y_2)-f(x, y_1)|\leqq g(x)|y_2-y_1|$, where $g(x)$ and $f(x, 0)$ are in $L(0, \infty)$. The special case: $f\equiv 0$, $(rp)'$ continuous and $\geqq 0$, was proved by W. Leighton [Proc. Nat. Acad. Sci. U. S. A. **35**, 190–191, 422 (1949); these Rev. **10**, 708]. *G. E. H. Reuter* (Manchester).

Home

***McLachlan, N. W. On a nonlinear differential equation in hydraulics.** Proceedings of Symposia in Applied Mathematics, Vol. V, Wave motion and vibration theory, pp. 49–61. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. $7.00.

The author treats the equation

$$\ddot{u}+2\kappa|u|\dot{u}+au=d \quad (a, \kappa>0, d\geqq 0)$$

which arises in various "square damping" problems. The one mentioned by the author is that of water flow in a hydroelectric system having a surge tank. He concludes that u always tends to d/a as t tends to infinity. He sketches the trajectories in the phase plane by the method of isoclines in three cases. For κ small he solves the equation by the method of Kryloff-Bogoliuboff.

The equation

$$\ddot{y}+2\kappa y\dot{y}+ay=0 \quad (a, \kappa>0)$$

is then discussed. This equation may be integrated once easily to get its phase-plane equation, which is graphed. Its time-dependent solution is gotten for y small by a perturbation development. *E. Pinney.*

Campbell, J. G., and Golomb, Michael. On the polynomial solutions of a Riccati equation. Amer. Math. Monthly **61**, 402–404 (1954).

Makai, E. A note on the solution of Heun's differential equation in a special case. Publ. Math. Debrecen **3** (1953), 140–143 (1954).

The author considers

$$w=y^{-b}(z-x)^{-b}{}_2F_1\left(a, b; c; \frac{z-y}{y}\,\frac{x}{z-x}\right)$$

as a function of x, y, z. Considered as a function of x alone, w satisfies a second-order ordinary linear differential equation of the Fuchsian class; the singular points of this equation are at $x=0, y, z, \infty$. Similar situations arise when w is considered as a function of y alone, or z alone. The author gives also a system of partial differential equations satisfied by w, and some other systems of differential equations with the property that the independent variable in one of them is a singular point of the others. *A. Erdélyi.*

Friedman, Bernard, and Mishoe, Luna Isaac. Eigenfunction expansions associated with a non-self-adjoint differential equation. Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, Res. Rep. No. **BR-4**, i+29 pp. (1954).

In solving certain boundary-value problems by the method of separation of variables, it is necessary to expand an arbitrary function in terms of the eigenfunctions of the ordinary differential equation $Au+\lambda Bu=0$, where A is a second-order operator, B a first-order operator. This paper is concerned with the special case

$$A=q(x)+\frac{d^2}{dx^2}, \quad B=p(x)-\frac{d}{dx}.$$

The theorem proved is as follows. Let $F(x)$ be of bounded variation in $0\leqq x\leqq 1$. Let $u_n(x)$ be the eigenfunction of $Au+\lambda Bu=0$ under the conditions $u(0)=u(1)=0$; and let $v_n(x)$ be the eigenfunctions of the adjoint system. Let $C(\lambda)e^{\lambda x}$ be the Wronskian of $Au+\lambda Bu=0$. Then, if

$$c\equiv F(0+)+\exp\left[-\int_0^1 p\,dt\right]F(1-)=0,$$

the series

$$\sum_{-\infty}^{\infty}u_n(x)\int_0^1 F(\xi)\frac{p(\xi)v_n(\xi)+v_n'(\xi)}{C'(\lambda_n)}d\xi$$

converges to $\frac{1}{2}\{F(x+0)+F(x-0)\}$, or to $F(x)$ at a point of continuity. But if $c\neq 0$, the series converges to

$$\tfrac{1}{2}\{F(x+0)+F(x-0)\}-\tfrac{1}{2}c\exp\left[\int_0^x p\,dt\right].$$

E. T. Copson (St. Andrews).

Kreĭn, M. G. On some cases of effective determination of the density of an inhomogeneous cord from its spectral function. Doklady Akad. Nauk SSSR (N.S.) **93**, 617–620 (1953). (Russian)

Under certain conditions the boundary-value problem $y''+q(x)y+\lambda y=0$, $ay(0)+by'(0)=0$ on an interval $0\leqq x<L$ $(L\leqq\infty)$ determines a unique spectral function. The questions considered here are: (1) what conditions on a non-decreasing function τ are necessary and sufficient in order that it be a spectral function of a differential problem as above, and (2) given a spectral function how can the function q, and constants a, b be explicitly recovered? These problems are formulated in the slightly more general form of the equation

$$y(x)=y(0)+y'(-0)x-\lambda\int_0^x(x-s)y(s)dM(s),$$

where $0\leqq x<L$ and $M(x)$ is interpreted as the mass of a vibrating string S on the interval $[0, x)$. Let φ, ψ be the solutions of the integral equation satisfying the initial conditions $y(0)=1$, $y'(-0)=0$ and $y(0)=0$, $y'(-0)=1$ respectively. The principal spectral function τ of S is the non-decreasing function on $0\leqq t<\infty$, satisfying $\tau(0)=0$,

$\tau(t)=\tau(t-0)$, and

$$\lim_{z\to L}\frac{\psi(x,\lambda)}{\varphi(x,\lambda)}=\int_0^{\infty}\frac{d\tau(t)}{t-\lambda}\quad(\lambda \text{ non-}\varepsilon\ (0,\infty))$$

[cf. Krein, same Doklady (N.S.) 87, 881–884 (1952); these Rev. 14, 868]. The first result is that a non-decreasing function τ on $0\leqq t<\infty$, $\tau(0)=0$ (non-negative spectrum) is a principal spectral function of a string S if and only if $\int_0^\infty(1+t)^{-1}d\tau(t)<\infty$, and the mass distribution M is uniquely determined by τ. A second result gives various rules of comparison between strings S and S^* (that is their lengths L, L^*, and their masses M, M^*) provided one knows certain relations between their spectral functions τ and τ^*. These results are illustrated by constructing the string with spectral density

$$\frac{d\tau}{dt}=\frac{P(t)}{\pi t^{1/2}Q(t)}\quad(0\leqq t<\infty),$$

where Q is a polynomial which is positive for $t\geqq 0$, and P is a polynomial with real non-negative zeros and whose degree is less than or equal to that of Q. Various examples are given.

E. A. Coddington (Los Angeles, Calif.).

Berezanskiĭ, Yu. M. On hypercomplex systems constructed from a Sturm-Liouville equation on a semi-axis. Doklady Akad. Nauk SSSR (N.S.) 91, 1245–1248 (1953). (Russian)

Let q be a function of bounded variation on $0\leqq t<\infty$, and $\tilde q$ any non-negative non-increasing function satisfying

$$|q(t_1)-q(t_2)|\leqq\tilde q(t_1)-\tilde q(t_2),\quad 0\leqq t_1<t_2<\infty;$$

in particular, $\tilde q(t)$ may be taken as the variation of q from t to ∞. Let μ be the solution of the equation $y''-\tilde q(t)y=0$ satisfying $\mu(0)=1$, $\mu'(0)=0$. It is non-decreasing and $\mu(t)\to+\infty$ if $\tilde q$ is not the zero function. Let q be extended to $-\infty<t<\infty$ by requiring it to be even. The translation operator T_s is defined for functions $x\in C^2$ on $0\leqq t<\infty$ by $(T_sx)(t)=u(t,s)$ $(0\leqq t,s<\infty)$ where u is the solution of the equation $u_{tt}-q(t)u=u_{ss}-q(s)u$ satisfying $u(t,0)=x(t)$, $u_s(t,0)=0$ $(x(-t)=x(t),t\geqq 0)$. If $L_1(0,\infty;\mu)$ denotes the L_1 space of functions on $0\leqq t<\infty$ with measure $\mu(t)dt$ and with multiplication defined by $(x*y)(t)=\int_0^\infty(T_sx)(t)y(s)ds$, it is stated that $L_1(0,\infty;\mu)$ is a semi-simple hypercomplex system with basis $[0,\infty)$ without unit. [For the definition of a hypercomplex system see the author, same Doklady (N.S.) 85, 9–12 (1952); these Rev. 14, 162.] A character of $L_1(0,\infty;\mu)$ is a bounded function χ on $0\leqq t<\infty$ such that $(T_s\mu\chi)(t)=\mu(t)\chi(t)\mu(s)\chi(s)$; and the characters are those functions χ_λ of the form $\chi_\lambda(t)=\varphi(t,\lambda)/\mu(t)$, where φ is the solution of $y''-q(t)y=-\lambda y$, $y(0)=1$, $y'(0)=0$, and λ is a complex number such that $\varphi(t,\lambda)=O(\mu(t))$. If $\tilde q(t)\to 0$, $t\to\infty$, then the characters may be identified with the spectrum of $y''-q(t)y=-\lambda y$, $y'(0)=0$. Using the above information and general results from the theory of hypercomplex systems and normed rings, the author states that the analogue of the Wiener theorem is valid if $\tilde q(t)\to 0$, $t\to\infty$: necessary and sufficient in order that the subspace spanned by the translates of functions $x\in M\subset L_1(0,\infty;\mu)$ be $L_1(0,\infty;\mu)$ is that the Fourier transforms

$$\hat x(\lambda)=\int_0^\infty x(t)\varphi(t,\lambda)dt,\quad x\in M,$$

have no common zero λ in the spectrum of $y''-q(t)y=-\lambda y$, $y'(0)=0$. The case $q(t)=O(t^{-\alpha-\epsilon})$, $\alpha=2,3$, $\epsilon>0$, was considered by Agranovich [ibid. 66, 1025–1028 (1949); these Rev. 11, 28]. *E. A. Coddington* (Los Angeles, Calif.).

Berezanskiĭ, Yu. M. On the uniqueness of the determination of Schrödinger's equation from its spectral function. Doklady Akad. Nauk SSSR (N.S.) 93, 591–594 (1953). (Russian)

Let L denote the partial differential operator given by $Lu=-\Delta u+c(x,y)u$, where c is a real piecewise-analytic function defined in the half-plane $x\geqq 0$. The function c is said to be piecewise-analytic if it is of class C^2 and it is possible to cover the half-plane with compact connected sets R_j (with bounded subsets of $x\geqq 0$ covered by finitely many R_j) in each of which c is analytic. Let A be the operator in L^2 of the half-plane $x\geqq 0$ having as domain the set of all $f\in C^2$ vanishing outside compact sets which satisfy $f_x(0,y)=0$, $-\infty<y<\infty$, and is defined by $Af=Lf$. If $E(\lambda)$ is the resolution of the identity corresponding to a self-adjoint extension of A [Reviewer's note: it appears that further restrictions on c must be made to assure the uniqueness of such an extension], then

$$(E\Delta)f(p)=\int\theta(p,q,\Delta)f(q)dq,$$

where p, q are points in the half-plane,

$$\Delta=(\lambda_1,\lambda_2],\quad E(\Delta)=E(\lambda_2)-E(\lambda_1),$$
$$\theta(p,q,\Delta)=\theta(p,q,\lambda_2)-\theta(p,q,\lambda_1),$$

where $\theta(p,q,\lambda)$ is the spectral function. The spectral kernel t is defined by

$$t(\alpha,\beta,\lambda)=\theta((0,\alpha),(0,\beta),\lambda),\ -\infty<\alpha,\beta,\lambda<\infty.$$

It is shown that if t_1, t_2 are spectral kernels corresponding to two operators A_1, A_2 with functions c_1, c_2 respectively, and if there exists an interval $I=(\alpha_0-\epsilon,\alpha_0+\epsilon)$, $\epsilon>0$, such that $t_1(\alpha,\beta,\lambda)=t_2(\alpha,\beta,\lambda)$ for α, $\beta\in I$ and $-\infty<\lambda<\infty$, then $c_1=c_2$ in the half-plane $x\geqq 0$. The proof results by exploiting results of M. Riesz on hyperbolic equations.

E. A. Coddington (Los Angeles, Calif.).

Levitan, B. M. On expansion in eigenfunctions of the equation $\Delta u+\{\lambda-q(x_1,x_2,x_3)\}u=0$. Doklady Akad. Nauk SSSR (N.S.) 94, 179–182 (1954). (Russian)

Let q be a continuous real function defined on a finite simply-connected region D plus its boundary Γ in three-dimensional euclidean space E_3. The eigenvalue problem $-\Delta u+qu=\lambda u$, $\partial u/\partial n=0$ on Γ, is considered. If it has positive spectrum $\lambda_k=\mu_k^2$ with eigenfunctions ω_k, the spectral function θ is defined by

$$\theta(P,Q;\mu)=\sum_{\mu_k<\mu}\omega_k(P)\omega_k(Q),$$

where $P,Q\in D$. Several results are stated which describe the asymptotic behavior of θ as $\mu\to\infty$, and these can be used to prove the existence of a spectral function in all of E_3 which satisfies the same asymptotic relations. Next it is stated that (under certain conditions) the difference between the Riesz mean of order one of the expansion of a function in eigenfunctions of the given problem on $D\cup\Gamma$ and the Riesz mean of order one of the ordinary Fourier expansion tends to zero as $\mu\to\infty$. [Reviewer's note: It appears that either the formula just following (6) of the paper should read $\lim R_1(P;\mu)=0$, $\mu\to\infty$, or else "first order" in the statement of the result should read "order s".] This result can be carried over to the whole space E_3. Analogous results are valid for E_2, and a result on the Riesz mean of order $\frac{1}{2}$ is stated to be an improvement of one due to Titchmarsh [Proc. London Math. Soc. (3) 3, 80–98 (1953); these Rev. 15, 229]. It is claimed the results are

valid for a general self-adjoint differential operator

$$\sum \frac{\partial}{\partial x_i}\left(a_{ij}(P)\frac{\partial u}{\partial x_j}\right)+(C(P)+\lambda)u=0$$

in E_2. No proofs are given *E. A. Coddington.*

Elianu, I. P. Les systèmes dérivés des systèmes différentiels extérieurs. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 4, 815–828 (1952). (Romanian. Russian and French summaries)

First, three theorems are proved concerning systems of homogeneous exterior forms. The derived system Ω' of a system Ω is defined as the system formed from the equations of Ω or of a system algebraically equivalent to Ω, such that its exterior differentials all belong to the Cartan ring of Ω. In the same way the series $\Omega'', \cdots, \Omega^{(m)}, \Omega^{(m+1)}=0$ can be constructed. $\Omega^{(m)}$ is the greatest closed system contained in Ω and this system can be used for the construction of all integral manifolds of Ω. *J. A. Schouten* (Epe).

Papy, Georges. Sur la réciproque du théorème de Volterra-Poincaré pour les formes à coefficients continus. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 40, 25–28 (1954).

A new proof is given of the theorem that every closed differential form is locally a derived form. The method uses the fact that $d\omega=BAB^{-1}\omega$ where A is the operator which multiplies a form on the left by $dx^1+\cdots+dx^n$ and B is defined by

$$B(a_{i_1\cdots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p})=\frac{\partial^p(a_{i_1\cdots i_p})}{\partial x^{i_1}\cdots\partial x^{i_p}}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

C. B. Allendoerfer (Seattle, Wash.).

Vekua, I. N. A boundary problem with oblique derivative for an equation of elliptic type. Doklady Akad. Nauk SSSR (N.S.) 92, 1113–1116 (1953). (Russian)

The author extends results previously obtained [Mat. Sbornik N.S. 31(73), 217–314 (1952); these Rev. 15, 230] to include the case $\Delta w+aw_x+bw_y+\lambda cw=f$ with the boundary conditions $\alpha w_x+\beta w_y+\mu\gamma w=\delta$. By a change of variables this is transformed to the equation $V_{\bar z}+B(z)\bar V+\lambda C(z)w=f_0$ with the condition $\mathrm{Re}[z^{-n}V]+\mu\gamma_0 w=\delta_0$. The integer n is called the index of the problem (see above reference for definitions). This latter problem is shown to be equivalent to the problem of solving a certain integral equation of Fredholm type. Then it is stated that if $n\geqq 0$, and if p is the number of solutions of the equivalent homogeneous integral equation it follows that the homogeneous differential problem ($f_0=\delta_0=0$) has $N=2n+2+p-r$ solutions where $0\leqq r\leqq\min(p, 2n+2)$. Additional results are stated for $n<0$; more specific statements can be made when the parameters λ, μ are sufficiently small. *M. H. Protter.*

***Gillis, P. Sur certaines classes d'équations aux dérivées partielles du second ordre, non linéaires.** Premier colloque sur les équations aux dérivées partielles, Louvain, 1953. pp. 105–118. Georges Thone, Liège; Masson & Cie, Paris, 1954.

Let $z=z(x_1, \cdots, x_{2n})$ be of class C^2 and form

$$\omega=(-z_2dx_1+z_1dx_2)+(-z_4dx_3+z_3dx_4)+\cdots +(-z_{2n}dx_{2n-1}+z_{2n-1}dx_n),$$

$$d\omega=(z_{11}+z_{22})dx_1dx_2+(-z_{14}+z_{23})dx_1dx_3+\cdots +(z_{2n-1,2n-1}+z_{2n,2n})dx_{2n-1}dx_{2n},$$

where $z_j=\partial z/\partial x_j$, $z_{kj}=\partial^2 z/\partial x_k\partial x_j$. Using the rule

$$dx_idx_j=-dx_jdx_i,$$

let $L(z)$ be defined by $(d\omega)^n=n!L(z)dx_1dx_2\cdots dx_{2n}$. In terms of the new variables $X_j=x_{2j-1}+x_{2j}\sqrt{-1}$ the function $L(z)$ can be written

$$L(z)=4^n\left|\frac{\partial^2 z}{\partial X_j\partial\bar X_k}\right|,\quad \bar X_k=x_{k-1}-x_{2k}\sqrt{-1}.$$

The differential equation $L(z)=a(x_1, \cdots, x_{2n})$ will be said to be of elliptic type if the Hermitian form

$$(\partial^2 z/\partial X_j\partial\bar X_k)\xi_j\bar\xi_k$$

is definite (positive or negative) for each solution z. The problem of Dirichlet for equations $L(z)=a$ of elliptic type is shown to have at most two (one) solutions for n even (odd). The last part of the paper discusses the results in the literature on the analytic character of solutions of elliptic equations. *F. G. Dressel* (Durham, N. C.).

Višik, M. I. On the first boundary problem for elliptic equations degenerating on the boundary of a region. Doklady Akad. Nauk SSSR (N.S.) 93, 9–12 (1953). (Russian)

Višik, M. I. On boundary problems for elliptic equations degenerating on the boundary of a region. Doklady Akad. Nauk SSSR (N.S.) 93, 225–228 (1953). (Russian)

Suppose the equation

$$Lu\equiv\sum_{i,k=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial u}{\partial x_k}\right)+\sum_{i=1}^{n}b_i\frac{\partial u}{\partial x_i}+c(x)u=h(x)$$

is elliptic in the half-space $x_n>0$ and parabolic on the hyperplane $x_n=0$. Let D be a domain in the half-space $x_n>0$ with part of its boundary Γ in the hyperplane $x_n=0$. Conditions on the coefficients are given which guarantee existence and uniqueness for the first and second boundary-value problems. The conditions concern the behavior of the a_{ik} and in particular the quantity a_{nn}. In addition the coefficients of the first order terms and the first derivatives of these coefficients are required to satisfy certain inequalities.

M. H. Protter (Berkeley, Calif.).

Pleijel, Åke. A study of certain Green's functions with applications in the theory of vibrating membranes. Ark. Mat. 2, 553–569 (1954). Please see the note p. 1140

For a domain in $E_2:(x_1, x_2)$ with sufficiently smooth boundary the author considers the boundary-value problem

$$\Delta^\omega u+\lambda u=0 \tag{1}$$

for the Neumann or Dirichlet case, where

$$\Delta^\omega u\equiv\partial^2 u/\partial x_1^2+\partial^2 u/\partial x_2^2-\omega u,$$

and ω is any fixed number $\geqq 0$. For the operator $\Delta^\omega u$ itself the author introduces and analyzes the Green's function especially in its dependence on ω and for $\omega\to\infty$, by starting out from the elementary solution $(2\pi)^{-1}K_0(\omega^{1/2}r)$ and constructing the compensating addend required. Also, on denoting the eigenvalues and eigenfunctions of (1) by $\lambda_\nu(\omega)$ and $\phi_\nu(x,\omega)$ the author establishes for each $\omega\geqq 0$ the relation, previously had for $\omega=0$,

$$\sum_{\lambda_\nu<t}(\varphi_\nu(x;\omega))^2\sim\frac{t}{2\pi},\quad t\to\infty$$

and

$$\sum\lambda_\nu(\omega)^{-s}=\frac{a_0}{s-1}+\sum_{\nu=1}^{k}\frac{a_{2\nu-1}(\omega)}{s+\frac{1}{2}(2\nu-3)}+\chi_k(s;\omega) \tag{2}$$

for any $k \geqq 1$, where $\chi_k(s;\omega)$ is analytic in $\mathrm{Re}(s) > -\frac{1}{2}(2k-1)$, and a_1 and a_2 are independent of ω.

[Remark of the reviewer. Relation (2) for $\omega > 0$ could be easily reduced to the case $\omega = 0$ by utilizing the "asymptotic" expansion

$$(\lambda_\nu + \omega)^{-s} \sim \sum_{p=0}^{\infty} \binom{-s}{p} \omega^p \lambda_\nu^{-s-p},$$

and this is a device which has already been fully described in Landau's Handbuch der Lehre von der Verteilung der Primzahlen [2 vols., Teubner, Leipzig-Berlin, 1909]. For a recent application of this universal device see the reviewer, Ann. of Math. (2) 57, 32–56 (1953); these Rev. 14, 986.]

S. Bochner (Princeton, N. J.).

Duff, G. F. D. A tensor equation of elliptic type. Canadian J. Math. 5, 524–535 (1953).

Le but de l'auteur est l'étude, sur une variété M à bord B, riemannienne, orientable, de classe C^∞, des problèmes de Dirichlet et Neumann pour l'équation (E_A) $\Delta\varphi + A\varphi = 0$, où φ est une p-forme, Δ le laplacien au sens de G. de Rham et A un tenseur d'ordre $2p$ définissant, par produit intérieur, un opérateur linéaire sur les p-formes. Cette étude généralise une étude antérièure [même J. 5, 196–210 (1953); ces Rev. 14, 903] relative à l'équation de Laplace; $t\varphi$ est la forme induite sur B par la p-forme φ définie sur M et $n\varphi = \varphi - t\varphi$. Le cas où A est défini positif est particulièrement envisagé. Pour toute solution de (E_A), on a

$$-\tfrac{1}{2}\Delta[\varphi^2] = \nabla^i \varphi^{(i_1 \cdots i_p)} \nabla_i \varphi_{(i_1 \cdots i_p)} + \langle (A+C)\varphi, \varphi \rangle$$

où $\langle\ \rangle$ désigne le produit scalaire local et C un tenseur du type de A construit à l'aide du tenseur de courbure [Lichnerowicz, Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 2, Amer. Math. Soc., Providence, R. I., 1952, pp. 216–223; ces Rev. 13, 492]. Si $A+C$ est défini positif, on en déduit que φ^2 n'admet pas de maximum à l'intérieur d'un domaine dans lequel φ est solution de (E_A). L'auteur construit alors, en modifiant légèrement la méthode de G. de Rham, une forme de Green pour (E_A) sur une variété close F. Sur F, l'équation (E) a au plus un nombre fini de solutions linéairement indépendantes, seulement la solution 0 si A (ou $A+C$) est défini positif. Dans ce cas il existe une solution fondamentale globale pour (E_A) sur F.

D'autre part, étant donnée une variété compacte M à bord B, on peut définir une variété close C^∞ (sur double) F dont elle est sous-variété et prolonger A sur F. A l'aide de cette technique, l'auteur étudie les problèmes de Dirichlet-Neumann d'abord pour A défini positif: le problème de Dirichlet admet une solution unique où $t\varphi$, $t{*}\varphi$ ont des valeurs données sur B; de même pour le problème de Neumann (valeurs données de $t{*}d\varphi$ et $t{*}d{*}\varphi$ sur B). Il en déduit, dans le cas général, des conditions nécessaires et suffisantes de possibilité en termes des "formes propres" des deux problèmes. C'est ainsi que le problème de Dirichlet correspondant à $t\varphi = t\psi$, $t{*}\varphi = t{*}\psi$ admet une solution si

$$\int_B (\psi \wedge {*}d\varphi_r - \delta\varphi_r \wedge {*}\psi) = 0$$

pour toute solution φ_r du problème correspondant à $t\varphi = 0$, $t{*}\varphi = 0$. On peut déduire de l'étude faite que (E_0) $(\Delta\varphi = 0)$ admet une singularité fondamental globale sur M si et seulement si le problème de Dirichlet pour (E_0) admet au plus une solution.

A. Lichnerowicz (Paris).

Watanabe, Yoshikatsu, and Nakamura, Mikio. On the partial differential equation of parabolic type with constant coefficients. J. Gakugei, Tokushima Univ. (Nat. Sci.) 4, 39–44 (1954).

The solution $\nu = \nu(t, \xi)$ of the parabolic equation

$$\nu_{tt} + 2h\nu_t + \nu = 0$$

subject to the conditions $\nu(0, \xi) = f(\xi)$, $\nu_t(0, \xi) = g(\xi)$ is obtained by letting $\epsilon \to 0$ in the solution of the corresponding boundary-value problem for the hyperbolic equation $\nu_{tt} - \epsilon^2 \nu_{\xi\xi} + 2h\nu + \nu = 0$. Here h, ϵ are constants, and the functions f, g are assumed to have Taylor's expansions in the vicinity of $\xi = 0$.

F. G. Dressel (Durham, N. C.).

Narasimhan, R. On the asymptotic stability of solutions of parabolic differential equations. J. Rational Mech. Anal. 3, 303–313 (1954).

Consider the nonlinear parabolic equation,

$$u = L(u) + F(x, t, u),$$

where $L(u)$ is an elliptic operator,

$$L(u) = \sum_{i,j} a_{ij}(x)\partial^2 u/\partial x_i \partial x_j + \sum_i b_i(x)\partial u/\partial x_i,$$

with the condition that u vanish on the boundary of a bounded, closed, convex domain G for $t \geqq 0$. Let F be a nonlinear function of u with the property that $uF(x, t, u) \leqq \lambda u^2$ for $x \in G$ and $t > 0$, provided that $|u| \leqq K$, a given constant, where λ is a given constant depending upon the domain G and the coefficients above. It is shown that the solution $u = 0$ is stable, which is to say that if $|u(x, 0)| \leqq \delta$ for $x \in G$, with δ sufficiently small, then $\lim_{t\to\infty} u(x, t) = 0$ uniformly in x. The method of proof combines the comparison theorems of Westphal and Prodi with the idea of the second method of Liapounoff for ordinary differential equations. Extensions of this result are given for systems. These results constitute generalizations of previous results of the reviewer (for rectangular regions) [Trans. Amer. Math. Soc. 64, 21–44 (1948); these Rev. 10, 43] and Prodi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 10, 365–370 (1951); these Rev. 13, 351].

R. Bellman (Santa Monica, Calif.).

Cameron, R. H. The generalized heat flow equation and a corresponding Poisson formula. Ann. of Math. (2) 59, 434–462 (1954).

The boundary-value problem

$$(1)\quad \begin{cases} G_{\xi\xi} - aG_t + \theta(t, \xi)G = 0 & \text{in } R: \\ & 0 < t < t_0 \leqq \infty,\ -\infty < \xi < \infty, \\ \lim_{t \downarrow 0} G(t, \xi) = \sigma(\xi) & \text{for almost all } \xi \end{cases}$$

is, for positive constant a, correlated with the Wiener integral

$$(2)\quad I_{\theta,\sigma}(t, \xi) = \int_C^w \exp\left\{ta^{-1}\int_0^1 \theta[t(1-s), 2(ta^{-1})^{\frac{1}{2}}x(s) + \xi]ds\right\} \times \sigma[2(ta^{-1})^{\frac{1}{2}}x(1) + \xi]d_w x,$$

integrated over the space C of continuous functions $x(s)$ on the interval $0 \leqq s \leqq 1$ satisfying $x(0) = 1$. Thus Theorem 1 gives the result: Let $\theta(t, \xi)$ be continuous in R and bounded in the neighbourhood of each point of the edge $t = 0$. Then if $G(t, \xi)$ is an almost-everywhere positive solution of (1) with continuous partial derivatives G_t and G_ξ in R, it follows that (3) $G(t, \xi) \geqq I_{\theta,\sigma}(t, \xi)$ in R. It is shown that under

certain conditions on θ and σ, a solution of (1) is given by (2). Uniqueness theorems are also discussed. By evaluating the Wiener integral (2) for the case $\theta=0$, the author indicates that (2) is an extension of the classical Poisson's formula for the heat equation. *K. Yosida* (Princeton, N. J.).

Kline, Morris. Asymptotic solution of linear hyperbolic partial differential equations. J. Rational Mech. Anal. 3, 315–342 (1954).

La solution $u(x, t)$ $[x\equiv(x_1, \cdots, x_{n-1}), t\equiv x_n]$ de l'équation aux dérivées partielles

$$Lu=\sum_{i,j=1}^{n} a^{ij}(x)\partial^2u/\partial x_i\partial x_j + \sum_{k=1}^{n} b^k(x)\partial u/\partial x_k+c(x)u=\partial^2f(x,t)/\partial t^2,$$

satisfaisant aux conditions initiales $u(x, 0)=\partial u(x, 0)/\partial t=0$, avec $f(x, t)\equiv y(x)h(t)H(t)$ ($H(t)$: fonction de Heaviside) s'exprime par la formule de Duhamel

$$u(x, t)=\frac{\partial}{\partial t}\int_0^t U_{[y]}(x, t-\tau)h(\tau)d\tau,$$

au moyen de la solution $U_{[y]}(x, t)$ de l'équation $Lu=g(x)\delta'(t)$ ($\delta(t)$: fonction de Dirac) avec les mêmes conditions initiales. L'auteur en déduit que si $h(t)\equiv\exp(-i\omega t)$, $u(x, t)$ prend la forme $u(x)\exp(-i\omega t)$ lorsque t est grand, avec

$$(*)\quad u(x)\sim\frac{g(x)}{a_{nn}}+\sum_\alpha[U(x, \tau_\alpha)]e^{i\omega\tau_\alpha} -\left(\frac{1}{i\omega}\right)\sum_\alpha\left[\frac{\partial}{\partial\tau}U(x, \tau_\alpha)\right]e^{i\omega\tau_\alpha} +\left(\frac{1}{i\omega}\right)^2\sum_\alpha\left[\frac{\partial^2}{\partial\tau^2}U(x, \tau_\alpha)\right]e^{i\omega\tau_\alpha}-\cdots,$$

où les $\tau_\alpha(x)$ désignent les valeurs de τ pour lesquelles $U(x, \tau)$ ou une quelconque de ses dérivées par rapport à τ subit une discontinuité par saut brusque, le saut correspondant étant noté $[U(x, \tau)]$, etc, (l'auteur se limite au cas où $U(x, t)$ ne possède pas d'autres discontinuités). Les développements sont purement formels.

L'évaluation asymptotique (*) n'est évidemment utilisable que si on connaît les valeurs des discontinuités de $U(x, \tau)$ et de ses dérivées successives. L'auteur reprend alors la théorie de la propagation des discontinuités par saut brusque des solutions de l'équation $Lu=0$ [cf. J. Hadamard, Leçons sur la propagation des ondes et les équations de l'hydrodynamique, Hermann, Paris, 1903] par une méthode qui consiste à remplacer l'équation de propagation par une équation intégrale, méthode développant et justifiant un procédé formel déjà utilisé par H. Bremmer [Comm. Pure Appl. Math. 4, 419–426 (1951); ces Rev. 13, 605]. Cette théorie appliquée à $U(x, t)$ fournit les renseignements nécessaires pour expliciter l'évaluation asymptotique (*).

L'exposé concerne principalement le cas du milieu illimité; l'auteur indique succinctement comment l'adapter au cas d'un milieu présentant des obstacles. *H. G. Garnir.*

Serrin, J. B. A note on the wave equation. Proc. Amer. Math. Soc. 5, 307–308 (1954).

Moyennant des hypothèses assez compliquées, l'auteur obtient une généralisation pour l'équation des ondes d'une proposition de la théorie du potentiel affirmant l'égalité des moyennes d'une fonction harmonique régulière sur des ellipsoïdes homofocaux. *H. G. Garnir* (Liège).

Marziani, Marziano. Sull'applicazione del calcolo simbolico alle equazioni di propagazione in tre dimensioni. Ann. Univ. Ferrara. Sez. VII. (N.S.) 2, 111–116 (1953).

In this note the solution of the equation of damped waves in three dimensions, $\partial^2U/\partial t^2-\Delta U-kU=X(P, t)$, is obtained by using the Laplace transform with respect to the time-variable t. The solution is required to satisfy an initial condition $U=U_0$, $\partial U/\partial t=(\partial U/\partial t)_0$ when $t=0$ and P lies in a domain S, and a boundary condition, that U and its normal derivatives take given values on the boundary σ of S for all $t>0$. *E. T. Copson* (St. Andrews).

Integral Equations

Hellsten, Ulf. The reality of the eigenvalues of certain integral equations. Ark. Mat. 3, 79–87 (1954).

For the integral equation $\phi(x)=\lambda\int_0^{f(x)}K(x, y)\phi(y)dy$ with $K(x,y)=P(x)Q(y)$; $P(x)Q(x)\geqq0$ and integrable on $0\leqq x\leqq1$; $0\leqq f(x)\leqq1$ monotonic non-decreasing with $f(x-0)>x$, except possibly for $x=0$ or $x=1$, it is shown that the eigenvalues are real and positive by obtaining a form for the Fredholm determinant. The Fredholm determinant is also of genus zero. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Stanković, Bogoljub. Solution d'une équation intégrale homogène. Srpska Akad. Nauka. Zbornik Radova 35. Mat. Inst. 3, 95–106 (1953). (Serbo-Croatian. French summary)

The author investigates the integral equation

$$f(x)=\lambda(\pi x)^{-1/2}\int_0^\infty f(t)\exp\left(-\frac{t^2}{4x}\right)dt$$

by means of the Laplace transformation. He first shows that the Laplace transform, $\phi(s)$, of $f(t)$ must satisfy the functional equation $\phi(s)=s^{-1/2}\lambda\phi(s^{1/2})$; and then he uses the asymptotic theory of Laplace transforms to show that for functions $f(x)$ of regular growth [in the sense of Karamata, Mathematica, Cluj 4, 38–53 (1930)] the only solutions of the functional equation are $\phi(s)=s^{-1}(\log s)^\nu$, with $\lambda=2^\nu$. Without the growth condition, the general solution of the functional equation with $\lambda=2^\nu$ is

$$\phi(s)=\frac{(\log s)^\nu}{s}\omega\left(\frac{\log\log s}{\log 2}\right)$$

where $\omega(z+1)=\omega(z)$. *A. Erdélyi* (Pasadena, Calif.).

Kurth, Rudolf. Zur Schwarzschildschen Integralgleichung. Z. Astrophys. 31, 115–120 (1952).

The author points out that in most practical applications of the so-called fundamental equation of stellar statistics of Schwarzschild the problem is equivalent to solving the integral equation

$$g(x)=\int_0^\infty K(y)f(x+y)dy\quad(K(x)\geqq0, 0\leqq x<\infty),$$

for the unknown function of f in terms of the known functions $K(y)$ and $g(x)$, it being given that both f and g are non-negative in their ranges. By expanding f and g in terms of the functions

$$\psi_\nu(x)=e^{-\nu ax}\quad(x>0;\ a=\text{constant}>0;\ \nu=1, 2, 3, \cdots),$$

the author shows that the solution is unique. However, there exist infinitely many solutions which satisfy the equation to any assigned precision. The principal conclusion then is the well known one that when the functions K and g are known only numerically (and often only as histograms) any numerical artifice to dodge the essential indeterminacy of the solution is bound to be illusory. *S. Chandrasekhar.*

Wall, H. S. Concerning harmonic matrices. Arch. Math. 5, 160–167 (1954).

The author studies the Stieltjes integral equation (1) $M(x, y)=I+\int_x^y dF(s)\cdot M(s, y)$, where F is an n^2 matrix, or is a suitably bounded infinite matrix, whose elements are possibly complex-valued functions, continuous and of b.v. (bounded variation on every real interval). M is said to be a h.m. (harmonic matrix) if its elements are functions of ordered pairs x, y of real numbers, continuous and of b.v. in x for each y, and continuous in y for each x, while for each ordered triple (x, y, z) one has $M(x, y)\cdot M(y, z)=M(x, z)$, $M(x, x)=I$. Let H_n be the class of h.m.'s of n^2 elements; Φ_n is the class of matrices (of n^2 elements) of functions, continuous and of b.v., with $F(0)=0$. The equality (1) implies mutual correspondence of H_n, Φ_n (if $M\in H_n$, $F\in\Phi_n$ and (1) holds, then M, F mutually correspond). Let F correspond to the 2^2 h.m. M; with z complex and y real, there exists a segment j containing y, so that in j the only continuous solution of the Riccati-Stieltjes integral equation

$$w(x)=z+\int_y^x w^2 dF_{21}+\int_y^x w d[F_{22}-F_{11}]-\int_y^x dF_{12}$$

is $w(x)=[M_{11}(x, y)z+M_{12}(x, y)]/[M_{21}(x, y)z+M_{22}(x, y)]$. Results similar to the above are established for infinite, suitably bounded matrices. If F corresponds to the h.m. M and z is a complex number-vector, then the only continuous vector-function $f(=f_i, i=1, \cdots, n)$ satisfying $f(z)=z+\int_x^y dF(s)\cdot f(s)$ $(\equiv v)$ is $f(x)=M(x, y)\cdot z$; if, moreover, g is a vector-function, continuous and of b.v., then the only continuous vector-function f satisfying $f(x)=v+\int_x^y dg(s)$ is $M(x, y)\cdot z+\int_x^y M(x, s)\cdot dg(s)$.

W. J. Trjitzinsky (Urbana, Ill.).

Have

***Magnus, W. Infinite matrices associated with a diffraction problem.** Proceedings of Symposia in Applied Mathematics, Vol. V, Wave motion and vibration theory, pp. 71–74. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. $7.00.

Abstract of a paper in Quart. Appl. Math. 11, 77–86 (1953); these Rev. 14, 816.

Functional Analysis, Ergodic Theory

Williamson, J. H. Compact linear operators in linear topological spaces. J. London Math. Soc. 29, 149–156 (1954).

The theory of linear transformations of the form

$$T(x)=x-U(x),$$

where U is a completely continuous transformation on a linear space X to X was developed for linear normed complete spaces by F. Riesz, was extended to linear topological spaces containing a bounded vicinity by D. H. Hyers [Revista Ci., Lima 41, 555–574 (1939); these Rev. 1, 318] and linear locally convex topological spaces by J. Leray [Acta Sci. Math. Szeged 12, Pars B, 177–186 (1950); these Rev. 12, 32]. The present paper defines a compact linear operator U as one for which there exists a neighborhood N_0 such that $U(N_0)$ is relatively compact and points out that in the Leray paper the "locally convex" condition is not needed. This involves giving a direct proof of the Fredholm alternative theorem for the linear case, i.e. $x-U(x)$, has a continuous inverse unless $x-Ux=0$ has nonvanishing solutions. Underlying the proof is the idea that a compact operator may be in a sense "approximated" by operators of finite rank. It is noted that reference to a dual space may be avoided by proving that the spaces $T^{-r}(0)$ and $X/T^r(X)$ are isomorphic for all r. [Cf. T. H. Hildebrandt, Acta Math. 51, 311–318 (1928).] *T. H. Hildebrandt.*

Nevanlinna, Rolf. Bemerkung zur absoluten Analysis. Ann. Acad. Sci. Fennicae. Ser. A. I. Math-Phys. no. 169, 7 pp. (1954).

This paper begins with a statement of appreciation of the merits of calculus in normed linear spaces, as developed by Fréchet and his successors. The author thinks this calculus has not received as much attention in actual practice as it deserves. He stresses the advantages of the coordinate-and-dimension-free vector notation in setting forth the essential "intuitive" geometric aspects of problems, and the fact that it leads to greater generality, unity, and transparency; it is, in addition, suitable for the framing of hypotheses which most naturally correspond to the natures of the problems posed. To illustrate this thesis he discusses the solution of the 'total differential' equation $dy=A(x)dx$, i.e. the finding of a function $y=f(x)$ whose Fréchet differential is (in a conventional notation) $df(x; dx)=A(x)dx$. Here the linear operator $A(x)$ is assumed to be differentiable and to satisfy the natural integrability condition (symmetry of the second differential of f). The treatment is notably simple and clear, and is characterized by at least one gain in generality over previous treatments, namely, continuity of the differential of $A(x)$ is not required. *A. E. Taylor.*

Shimoda, Isae. Notes on general analysis. III. On the norm of analytic functions. J. Gakugei, Tokushima Univ. (Nat. Sci.) 4, 1–10 (1954).

[For parts I–II see same J. 2, 13–20 (1952); 3, 12–15 (1953); these Rev. 14, 766; 15, 38.] Two examples are given of the following situation: f is analytic and not constant in a sphere $\|x\|<R$ in a complex Banach space, with values in a second such space, but $\|f(x)\|$ is constant when $\|x\|<R$. That this situation can occur even when the space of the variable x is the ordinary complex plane has been known for some time. [See the example in the reviewer's paper, Math. Mag. 23, 115–124 (1950), p. 118; these Rev. 11, 507.] The maximum-modulus theorem holds in the following form: if f is analytic in the connected open set D, if $\|f(x)\|$ is not constant in D and $\|f(x)\|\leqq M$ in D, then $\|f(x)\|=M$ cannot occur in D. If f is analytic in a neighborhood of 0, then for sufficiently small positive r, $M(r)=\sup_{\|x\|=r}\|f(x)\|$ is finite. The author generalizes the classical theorems about the properties of $M(r)$. An example is cited in which f is analytic for all x, $M(r)=\frac{1}{2}$ if $0<r<\frac{1}{2}$, $M(r)$ is strictly increasing when $\frac{1}{2}\leqq r<1$, $M(r)\to\infty$ as $r\to 1$.

A. E. Taylor (Los Angeles, Calif.).

Nečepurenko, M. I. On Čebyšev's method for functional equations. Uspehi Matem. Nauk (N.S.) 9, no. 2(60), 163–170 (1954). (Russian)

Let ϕ be a mapping of a Banach space X into a Banach space Y which possesses a second Fréchet differential ϕ''.

The Čebyšev method attempts to solve the equation $\phi(x)=0$ by means of the iteration

$$x_{n+1}=x_n-\Gamma_n\phi(x_n)-\tfrac{1}{2}\Gamma_n\phi''(x_n)[\Gamma_n\phi(x_n)]^2,$$

where $\Gamma_n=[\phi'(x_n)]^{-1}$. The approach is close to a discussion of the Newton method made by Kantorovič [Doklady Akad. Nauk SSSR (N.S.) **59**, 1237–1240 (1948); these Rev. **9**, 537]. Under assumptions very similar to the paper cited [which involve a good initial guess x_0, the existence of a bounded inverse for the operator $\phi'(x_0)$, and a bound for the norm of the bilinear operator ϕ'' near x_0], the existence of a locally unique solution x^* is assured and it is shown that $\|x_n-x^*\|$ vanishes like $(\frac{3}{4})^n A^{3^n-1}$, $0<A<1$. The author asserts that this method is at least as good as Newton's and compares favorably in practice with the method of tangent hyperbolas discussed by Salehov [ibid. **82**, 525–528 (1952); these Rev. **14**, 91] and Mertvecova [ibid. **88**, 611–614 (1953); these Rev. **15**, 39] without some of the computational disadvantages of this latter method. Čebyšev's method is compared with Newton's on a non-linear integral equation and with the tangent hyperbola method on two numerical equations. *R. G. Bartle* (New Haven, Conn.).

Kadec, M. I. On conditionally convergent series in the space L^p. Uspehi Matem. Nauk (N.S.) **9**, no. 1(59), 107–109 (1954). (Russian)

Extensions (of a restricted kind) of the classical results of Steinitz [J. Reine Angew. Math. **143**, 128–175 (1913), p. 144] on conditionally convergent series of vectors, from finite-dimensional spaces to the spaces L^p, $1<p<\infty$, are obtained. The principal results are as follows. 1. Let $v_i^1\in L^p$, $i=1,\cdots,N$, $\sum_1^N v_i^1=0$. There is a renumeration of the v_i^1, say v_i, for which

$$\max_k\left\|\sum_1^k v_i\right\|<\begin{cases}B_p\left\{\sum_1^N\|v_i\|^2\right\}^{1/2} & (p>2),\\ B_p\left\{\sum_1^N\|v_i\|^p\right\}^{1/p} & (1<p\leqq 2),\end{cases}$$

where B_p depends only on p. (For the proof elementary inequalities of Hölder type are employed.) 2. Assume either

$$\sum_1^\infty\|v_i\|^2<\infty\quad(p>2)$$

or

$$\sum_1^\infty\|v_i\|^p<\infty\quad(1<p\leqq 2),$$

where $v_i\in L^p$. Then: (a) if v is the limit of a subsequence of partial sums of $\sum_1^\infty v_i$, then v is in the sum region of $\sum_1^\infty v_i$, i.e., there is a rearrangement v_i^1 such that $v=\sum_1^\infty v_i^1$; (b) the sum region of $\sum_1^\infty v_i$ is a closed linear set. Proofs are analogs of those given previously by the author [Uspehi Matem. Nauk (N.S.) **8**, no. 1(53), 139–143 (1953); these Rev. **14**, 866]. *B. Gelbaum* (Minneapolis, Minn.).

Hewitt, Edwin. Fourier transforms of the class $\mathfrak{L}_p$. Ark. Mat. **2**, 571–574 (1954).

Continuing in the direction indicated by recent notes of I. E. Segal [Acta Sci. Math. Szeged **12**, Pars B, 157–161 (1950); these Rev. **12**, 188, 1002], the author [Proc. Amer. Math. Soc. **4**, 663–670 (1953); these Rev. **15**, 119] and the reviewer [ibid. **5**, 71–78 (1954); these Rev. **15**, 633], the present paper solves a question raised by Segal [loc. cit.]. Let G be a locally compact abelian group, G^* its dual, p an exponent satisfying $1<p\leqq 2$, p' the conjugate exponent defined by $1/p+1/p'=1$; $\mathfrak{L}_p(G)$ and $\mathfrak{L}_{p'}(G^*)$ are the usual Lebesgue spaces (relative to Haar measure on G and G^* respectively). T^p denotes the Fourier transformation mapping $\mathfrak{L}_p(G)$ into $\mathfrak{L}_{p'}(G^*)$. Theorem: If $1<p<2$ and G is infinite, the image $T_p(\mathfrak{L}_p(G))$ is a dense set of the first category in $\mathfrak{L}_{p'}(G^*)$, and the complementary set is dense and of the second category. The proof uses two lemmas about weakly convergent sequences in spaces $\mathfrak{L}_q(G)$, together with standard theorems about Banach spaces.

R. E. Edwards (London).

Nakamura, M., Takeda, Z., and Turumaru, T. On some extended principal axis theorems for completely continuous operators. Tôhoku Math. J. (2) **5**, 190–193 (1953).

Theorem 1 asserts that, if A is a completely continuous operator on a Hilbert space, there exist unitary operators U and V such that UAV is diagonal. This theorem is false, as may be seen by taking A to be the matrix which is 0 everywhere except for a sequence $1/i$ on the diagonal above the main diagonal. The theorem may be salvaged by a suitable supplementary hypothesis such as: the null spaces of A and A^* have equal dimension. The remaining theorems of the paper concern joint diagonalization of several operators, and may be rectified in the same way.

I. Kaplansky (Chicago, Ill.).

Orlov, S. A. On the deficiency index of linear differential operators. Doklady Akad. Nauk SSSR (N.S.) **92**, 483–486 (1953). (Russian)

Let l denote the linear quasi-differential operator defined by

$$l=p_n-D\{p_{n-1}D-D[p_{n-2}D^2-\cdots-D(p_1D^{n-1}-Dp_0D^n)]\},$$

where $D=d/dx$ and the p_k are real measurable functions on $0\leqq x<\infty$ satisfying for every finite b, $0<b<\infty$, the conditions

$$\int_0^b|p_0(x)|^{-1}dx<\infty,\quad\int_0^b|p_k(x)|dx<\infty,\quad k=1,2,\cdots,n.$$

By properly defining the domain of l it becomes a closed symmetric operator L_0 in $L_2(0,\infty)$. The deficiency index of L_0 is characterized in terms of the rank of a matrix involving the solutions of the equation $ly=\lambda y$. The proof follows from introducing a system via the "quasi-derivatives"

$$D^{[k]}=D^k\quad(k=0,1,\cdots,n-1),\quad D^{[n]}=p_0D^n,$$
$$D^{[n+k]}=p_kD^{n-k}-DD^{[n+k-1]}\quad(k=1,2,\cdots,n);$$

thus $l=D^{[2n]}$. A corollary is that the maximal number m of linearly independent solutions of $ly=\lambda y$ for non-real λ satisfies $n\leqq m\leqq 2n$, a result proved earlier by I. M. Glazman. If the equation $ly=\lambda y$ has a regular singular point at ∞, it is shown that m is the number of roots of a certain polynomial in a half-plane. There exist coefficients p_k for l which make ∞ a regular singular point for $ly=\lambda y$, and such that m is any desired integer satisfying $n\leqq m\leqq 2n$.

E. A. Coddington (Los Angeles, Calif.).

Šmul'yan, Yu. L. On holomorphic bounded matrix functions with determinant identically zero. Doklady Akad. Nauk SSSR (N.S.) **93**, 625–627 (1953). (Russian)

The author first gives without proof a result for an operator $w(\zeta)$ on a Hilbert space, where in the region G of the complex ζ-plane $w(\zeta)$ is finite-dimensional and holomorphic with norm $\leqq 1$. There is then a finite number r, the "functional rank" of $w(\zeta)$, which is the dimensionality of $w(\zeta)$ for

all $\zeta \in G$ except for a set with no limit-point in G; in addition, an explicit representation of $w(\zeta)$ is possible. Taking G to be $|\zeta|<1$, the factorisation $w(\zeta)=w_1(\zeta)w_2(\zeta)$ is shown to be possible, where $\|w_1(\zeta)\|\leqq 1$, $\|w_2(\zeta)\|\leqq 1$, and the range of $w_1(\zeta)$ is a fixed r-dimensional subspace. Assuming further that $w(\zeta)$ is an m-by-m matrix, and that $\det w(\zeta)\equiv 0$, $w_1(\zeta)$ and $w_2(\zeta)$ can be taken to be m-by-r and r-by-m matrices. The paper extends in part results of Potapov [same Doklady (N.S.) 72, 849–852 (1950); these Rev. 13, 736].

F. V. Atkinson (Ibadan).

Šmul'yan, Yu. L. Operators with degenerate characteristic functions. Doklady Akad. Nauk SSSR (N.S.) **93**, 985–988 (1953). (Russian)

Let V be an isometric operator in a Hilbert space $\mathfrak{H}$, defined on a space G and having a range G'; $D=\mathfrak{H}\ominus G$ and $D'=\mathfrak{H}\ominus G'$ are its deficiency spaces, and T is an "orthogonal extension" of V. The author first gives formulae concerning the "normed characteristic function" of T, which is an operator on D into D' given, under restrictions, by

$$w_H(T,\zeta)=|I-TT^*|^{-1/2}(T-\zeta I)(I-\zeta T^*)^{-1}|I-T^*T|^{1/2},$$

including a multiplication theorem for the case when T is formed from orthogonal extensions T_1, T_2 of isometric operators V_1, V_2 in two unitary spaces $\mathfrak{H}_1$, $\mathfrak{H}_2$ [cf. Livšic and Potapov, same Doklady (N.S.) 72, 625–628 (1950); these Rev. 11, 669]. This and results of the previously-reviewed paper are applied to prove, in outline, his "basic theorem 2" for an operator T, with $\|T\|\leqq 1$, which is an orthogonal extension of an isometric V with deficiency-indices (m, m) with $m<\infty$, such that every ζ with $|\zeta|<1$ is a characteristic value of T. Then T has an invariant subspace $\mathfrak{H}_1$, in which the induced operator T_1 is the orthogonal extension of an isometric V_1 with deficiency-indices (m, n) with $m>n$. Also given is a converse proposition, and a result on the vanishing of the characteristic function of a "simple" isometric V with arbitrary deficiency-indices not necessarily finite or equal.

F. V. Atkinson (Ibadan).

Davis, Philip, and Walsh, J. L. On representations and extensions of bounded linear functionals defined on classes of analytic functions. Trans. Amer. Math. Soc. **76**, 190–206 (1954).

Let $H(B)$ be a Hilbert space of complex-valued functions analytic in a region B. If the inner product is given by $(f,g)=\int\int_B f\bar{g}$ and all functions f with $\|f\|=(f,f)^{1/2}<\infty$ are admitted, the space is denoted by $L^2(B)$. If $\{\phi_n\}$ is a basis for $H(B)$, then a linear function L on $H(B)$ is continuous if and only if $\sum|L(\phi_n)|^2<\infty$. Then $H(B)$ is said to have a reproducing kernel [Aronszajn, same Trans. 68, 337–404 (1950); these Rev. 14, 479] if there is a function $K(z_1, z_2)$ such that $f(z)=(f(\cdot), K(\cdot, \bar{z}))$. The space $H(B)$ has a reproducing kernel if and only if the point functionals $L(f)=f(b)$, for $b \in B$, are all continuous. When $K(z, \bar{z})$ is continuous, the inequality $|f(z)|\leqq\|f\|K(z,\bar{z})^{1/2}$ shows that every linear functional on $H(B)$ which is continuous in the compact-open topology is also continuous in the norm topology; in particular, the functionals $L(f)=f^{(n)}(b)$ are also continuous. Any continuous functional has the representation $L(f)=(f, g)$. If L_n is a total set, the corresponding functions $g_n \in H(B)$ are complete so that L may be expressed as $\sum_0^\infty C_n\sum_1^n A_{nk}L_k$. Choosing $L_k(f)=f^{(k)}(0)$, it follows that $L(f)=\sum_1^\infty C_n(2\pi i)^{-1}\int_C R_n(z)f(z)\,dz$, where $\{R_n(z)\}$ is a sequence of rational functions. L is said to have a Cauchy representation if there is a function $C(z)$ regular outside C with $L(f)=(2\pi i)^{-1}\int_C C(z)f(z)dz$. Let G be a region whose closure lies in B. The authors consider the problem of extending a functional L, continuous on $L^2(B)$, so as to remain continuous on $L^2(G)$. Example: G a convex region, b a point of B exterior to G; here, $L^2(B)$ is dense in $L^2(G)$, but the functional $L(f)=f(b)$ cannot be extended to $L^2(G)$. A necessary and sufficient condition that a functional L be extendible to some $L^2(G)$ is that L have a Cauchy representation.

Let B be a region containing the origin, and bounded by a finite number of Jordan curves. Let G_r be the set of $z \in B$ with $g(z, 0)>-\log r$, where $g(z, 0)$ is the Green function for B. Let g_n be the representation functions for the functionals $L_k(f)=f^{(k)}(0)$, and let $\{\varphi_n\}$ be the orthogonal set obtained from $\{g_n\}$. Then if $\limsup|L(\varphi_n)|^{1/n}=r$, L may be extended to $L^2(G_{r'})$ when $r'>r$, but not when $r'<r$. When B is simply connected and ∂B analytic, it follows that a functional L can be extended to some $L^2(G)$ if and only if its representation function g can be continued analytically across ∂B everywhere.

R. C. Buck (Madison, Wis.).

Nikodým, Otton Martin. Sur les opérateurs normaux maximaux dans l'espace hilbertien séparable et complet. Notion de "lieu" et ses propriétés. C. R. Acad. Sci. Paris **238**, 1373–1375 (1954).

Nikodým, Otton Martin. Sur les opérateurs normaux maximaux dans l'espace hilbertien séparable et complet. Représentation canonique. II. C. R. Acad. Sci. Paris **238**, 1467–1469 (1954).

In these two short papers the author outlines the extension to normal operators of his results [Ann. Inst. H. Poincaré 11, 49–112 (1949); these Rev. 11, 670] on developing the multiplicity theory of self-adjoint operators from the concept of tribes of subspaces of Hilbert space.

F. H. Brownell (Seattle, Wash.).

Bonsall, F. F. A minimal property of the norm in some Banach algebras. J. London Math. Soc. **29**, 156–164 (1954).

It is proved that if A is a Banach algebra of bounded real-valued functions on a set X, with sup norm, or if A is a B^* algebra, and if N is another norm for A (in which A need not be complete) such that $N(a)\leqq\|a\|$ for all A, then $N=\|\ \|$. The same is also proved for $B^\#$ annihilator algebras A, where $B^\#$ means that for every a in A, there is a b in A such that $\|(ba)^n\|=\|a\|^n\|b\|^n$ for all n; and annihilator algebra means that $A^\perp={}^\perp A=(0)$, while for each proper closed left (right) ideal I, the ideal $I^\perp({}^\perp I)\neq(0)$. As in the paper of Kaplansky [Duke Math. J. 16, 399–418 (1949); these Rev. 11, 115], these norm conditions are used to establish a structure theory with the following result. Every $B^\#$ annihilator algebra is the $B(\infty)$ direct sum of minimal closed two-sided ideals A_α, which are themselves simple $B^\#$ annihilator algebras. The latter are identical with the closure of the algebra of operators of finite rank in a reflexive Banach space. These results are admittedly deducible from Kaplansky's work, but the author's approach is claimed to be simpler. However, for essential parts of the proof one is referred to a paper by the author and A. W. Goldie to appear later.

R. Arens (Princeton, N. J.).

Calculus of Variations

Transue, W. Sopra un teorema di Cinquini sull'esistenza dell'estremo in campi illimitati. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 333–336 (1952).

For a variational integral of the form

$$\int_{C^{[n]}} f(x, y(x), y'(x), \cdots, y^{(n)}(x))dx$$

the author establishes a theorem on the existence of an absolute minimum which extends earlier results of Cinquini [Ann. Scuola Norm. Super. Pisa (2) 6, 191–209 (1937)].
W. T. Reid (Evanston, Ill.).

Transue, William. Relazioni fra teoremi di esistenza del minimo in campi illimitati. Ann. Mat. Pura Appl. (4) 34, 411–419 (1953).

In a series of papers [Ann. Scuola Norm. Super. Pisa (2) 5, 169–190 (1936); 6, 191–209 (1937); Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 2(71), 211–218 (1938)], Cinquini established a series of existence theorems for calculus-of-variations problems for the integral

$$\int f[x, y(x), y'(x), \cdots, y^{(n)}(x)]dx.$$

The author, in the paper reviewed above, has also given such an existence theorem. In the present paper he studies relations between the hypotheses entering into these theorems, finding that some are special cases of others, etc.
J. M. Danskin (Washington, D. C.).

Föllinger, Otto. Diskontinuierliche Lösungen von Variationsproblemen mit Gefällebeschränkung. Math. Ann. 126, 466–480 (1953).

This paper treats the minimum problem for $\int f(x, y, y')dx$, where admissible curves are subject to the restriction $g(x, y) \leqq y' \leqq G(x, y)$. In the two cases considered, the minimizing curve is supposed to be composed of two smooth arcs E_0 and $\bar{E}_0$ meeting at an angle. In the first case E_0 satisfies $g(x, y) < y' < G(x, y)$; in the second, E_0 satisfies $y' = g(x, y)$. In both cases the second arc $\bar{E}_0$ satisfies $y' = G(x, y)$. Necessary conditions and sufficient conditions are given, but are too complicated to state in detail here. The special case where g and G are constants was previously treated by Flodin [Acta Soc. Sci. Fennicae. Nova Ser. A. 3, no. 10 (1945); these Rev. 7, 208]. The results of this paper are not included in the much more general treatments of Valentine [Contributions to the calculus of variations 1933–1937, Univ. of Chicago Press, 1937, pp. 403–447] and of Pennisi [Trans. Amer. Math. Soc. 74, 177–198 (1953); these Rev. 14, 661], since these authors assume the minimizing curve has no corners.
L. M. Graves (Chicago, Ill.).

On order

✶Dedecker, Paul. Equations différentielles extérieures et calcul des variations. Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 3, 14 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1953.

The author develops a theory of calculus of variations in several variables with exterior differential forms, based on the work of Lepage; the question has also been considered by Carathéodory, de Donder, Weyl. The theory is designed to parallel the one-dimensional case, but it is impossible to preserve the analogy completely. Let V_n be a real-analytic manifold. A C.V. problem of dimension m and differential order 0 is defined by an exterior differential m-form Ω_m. An m-manifold W_m in V_n is an extremal (roughly) if for every differentiable regular simplex Σ the first variation of $\int_\Sigma \Omega_m$, with the boundary of Σ held fixed, vanishes. The problem is bound if in addition there is given an ideal J of differential forms in V_n, and during the variation Σ is required to lie on integral manifolds of J. The basic space for the further discussion is the space $\Gamma_m{}^n$ of m-contact elements, a fiber space over V_n. A C.V. problem, on V_n, of order 1, is a bound C.V. problem of order 0 on $\Gamma_m{}^n$, such that the form Ω_m involves only the differentials of V_n and where the ideal J is determined once and for all (corresponding classically, for $n=1$, to the fact that the variable y' in the integrand means dy/dx). A Lagrangian function is defined. The equation of the extremals is obtained by considering Lagrange multipliers. This is first done locally, but later globally so as to obtain a fiber space over $\Gamma_m{}^n$, with fiber a Euclidean space, the coordinates in which are just the multipliers; this fiber space is called the phase extension of the problem. It is stated that all this can be considered as a generalization of the "invariant integral" of Cartan-Poincaré. The phase extension leads to a Hamiltonian formalism: canonical coordinates, moments, Hamiltonian function, energy equation, canonical equation. Finally indicatrix and figuratrix are considered briefly.
H. Samelson (Princeton, N. J.).

Bertolini, Fernando. Traiettorie luminose in uno spazio trasparente. Esistenza di regioni non illuminate da una sorgente luminosa puntiforme. Ricerche Mat. 2, 78–90 (1953).

The author establishes the existence of the absolute minimum for a variational integral of the form $\int_L f(P)ds$ in certain classes of "quasi-rectifiable" curves, where $f(P)$ is a function of position that is non-negative and lower semi-continuous in a closed region T of r-dimensional space with the possible exception of a finite number of points, and for which there is a non-negative continuous function $g(t)$ on $(0, \infty)$ that is non-summable and such that $f(P) \geqq g(|P-O|)$. To illustrate the significance of the latter condition the author presents an example of a plane problem in which $f(P)$ is a function of $|P-O|$ that is positive and of class C' throughout the plane, and such that if $|P_0-O|$ is suitably large the extremals emanating from P_0 do not penetrate certain associated regions. It is to be remarked that for such an illustration the author could have referred to an example given in Carathéodory's Variationsrechnung [Teubner, Leipzig-Berlin, 1935, p. 305] in which, however, $f(P)$ is not a function of the radius vector alone. The choice of the title of the paper is justified by a brief remark on the relation of the considered type of problem to geometrical optics.
W. T. Reid (Evanston, Ill.).

Bellman, Richard. Dynamic programming and a new formalism in the calculus of variations. Proc. Nat. Acad. Sci. U. S. A. 40, 231–235 (1954).

The first problem the author treats is the classical single-integral maximum problem with one endpoint fixed, for which he derives, but does not recognize, the classical transversality conditions [see, e.g., Morse, Calculus of variations in the large, Amer. Math. Soc. Colloq. Publ., v. 18, New York, 1934, p. 20]. He then turns to the eigenvalue problem derived from

$$\frac{d^2x}{dt^2} = -\lambda\phi(t)x, \quad x(0) = x(1) = 0,$$

and finds a partial differential equation from which one can calculate the eigenvalues.
J. M. Danskin.

Theory of Probability

Gonseth, F. Quatrièmes entretiens de Zurich. Conférence d'ouverture. Dialectica 7, 303–317 (1953).

Bernays, P. Einleitendes Referat. Dialectica 7, 318–321 (1953).

Jecklin, H. Einleitendes Referat. Dialectica 7, 322–325 (1953).

Nolfi, Padrot. Einleitendes Referat. Dialectica 7, 326–330 (1953).

Erismann, Th. Wahrscheinlichkeit im Sein und Denken. Dialectica 7, 331–346 (1953).

Nolfi, Padrot. Über Ordnung und Unordnung. Dialectica 7, 347–348 (1953).

The subject of these Entretiens was the foundations of probability theory and statistics.

de Finetti, Bruno. La notion de "distribution d'opinions" comme base d'un essai d'interpretation de la statistique. Publ. Inst. Statist. Univ. Paris 1, no. 2, 1–19 (1952).

The author discusses more concretely the ideas of a previous paper [Proc. 2nd Berkeley Symposium Math. Statistics and Probability, 1950, Univ. of California Press, 1951, pp. 217–225; these Rev. 13, 851]. The essential point is that not only are there "opinions", that is, prior distributions of hypotheses, to be considered, but also probability distributions of these opinions. *J. L. Doob.*

Matschinski, Matthias. Résultats d'observation et leurs probabilités a priori et a fortiori. C. R. Acad. Sci. Paris 238, 1861–1863 (1954).

Fréchet, Maurice. Une propriété générale des valeurs typiques d'un nombre aléatoire. Publ. Inst. Statist. Univ. Paris 1, no. 1, 1–48 (1952).

Let x be a random variable. Doss has proved that there is a number γ satisfying the inequality (*) $|\gamma-\lambda|\leqq T\{|x-\lambda|\}$ for all λ, if $T\{y\}=E\{y\}$. The author studies the problem of solving (*) for γ with various other interpretations of T. He considers (a) the central value (midpoint of the interval of variation), (b) median, (c) mode, (d) typical value of order α, that is, a value of a minimizing $E\{|y-a|^\alpha\}$. In some cases he finds a unique solution of (*), in others he exhibits counterexamples. For example, in case (d), if x is bounded and if $\alpha>1$, there is a unique typical mean of order α which is the unique solution of (*). If $\alpha>1$, and if $E\{|x|^{\alpha-1}\}<\infty$, the definition of typical mean of order α is generalized to be applicable, is unique, and is a solution of (*). *J. L. Doob* (Urbana, Ill.).

Hess, F. G. Alternative solution to the Ehrenfest problem. Amer. Math. Monthly 61, 323–328 (1954).

Le "modèle d'Ehrenfest" peut se décrire ainsi: $2R$ boules sont réparties entre 2 boites; à chaque instant . . . , $k-1, k, k+1, \cdots$, l'une des $2R$ boules est tirée au sort, avec équiprobabilité, et changée de boite; il pose le problème de calculer la probabilité $P(n|m;s)$ qu'à l'instant $(k+s)$ il y ait $(R+n)$ boules dans la première boite, conditionnelle quand il y en a m à l'instant k. Ce problème a été résolu par M. Kac [même Monthly 54, 369–391 (1947); ces Rev. 9, 46], et par A. J. F. Siegert [Physical Rev. (2) 76, 1708–1714 (1949)]; l'auteur donne une solution, voisine de celle de Siegert, en considérant la chaine de Markoff dont les états possibles sont les diverses répartitions possibles des $2R$ boules entre les 2 boites. *R. Fortet* (Paris).

Féron, R., et Fourgeaud, C. Sur le rapport de deux variables aléatoires. Publ. Inst. Statist. Univ. Paris 1, no. 2, 50–52 (1952).

The characteristic function of the quotient Y/X of two random variables with absolutely continuous distribution function is expressed in terms of the characteristic function of Y and the frequency function of X. A condition for the analyticity of the characteristic function of the quotient is given but is not established in a satisfactory manner. *E. Lukacs* (Washington, D. C.).

Féron, R., et Fourgeaud, C. Quelques propriétés caractéristiques de la loi de Laplace-Gauss. Publ. Inst. Statist. Univ. Paris 1, no. 2, 44–49 (1952).

The authors prove the following theorem: If $x=ay+Y$ (Y independent of y) and $y=bx+X$ (X independent of x) then x and y have a bivariate normal distribution. A generalization to the trivariate case is also given. The proof is analogous to a proof given by G. Darmois [C. R. Acad. Sci. Paris 232, 1999–2000 (1951); these Rev. 12, 839] for a related characterization of the normal distribution. *E. Lukacs* (Washington, D. C.).

Fortet, Robert, et Mourier, Edith. Sur les fonctionnelles de certaines fonctions aléatoires. C. R. Acad. Sci. Paris 238, 1557–1559 (1954).

Let $X_1, X_2, \cdots$ be mutually independent random variables with a common distribution, taking on values in a Banach space. Then, under specified conditions, if $Z_n=n^{-1/2}\sum_1^n X_j$, and if f is a function with specified properties, the distribution of $f(Z_n)$ is asymptotically that of $f(Y)$, where Y is Laplacian. Applications of this theorem are described, and it is extended slightly. *J. L. Doob.*

Finzel, Lothar. Untersuchungen über die wahrscheinliche Lage der Wurzeln reeller algebraischer Gleichungen. Math. Nachr. 11, 85–104 (1954).

In continuation of the work of Specht [Math. Nachr. 4, 126–149 (1951); these Rev. 13, 32] on complex polynomials, the author studies the probability that a real equation $x^n-a_1x^{n-1}+\cdots+(-1)^na_n=0$ have exactly m roots in a real interval R. Such equations, which may be represented by points $(a)=(a_1, \cdots, a_n)$ of n-dimensional space, are said to belong to class K if the locus K of point (a) is closed and bounded, has a positive n-dimensional content $J_n(K)$ and intersects every k-dimensional hyperplane:

$$c_{m0}+c_{m1}a_1+\cdots+c_{mn}a_n=0,$$

$m=1, 2, \cdots, n-k$, in a point set with a k-dimensional content. Let subclass K_p of K consist of the equations (a) in K with exactly p pairs of conjugate imaginary roots. Then $P_m(K)=J_n(K_p)/J_n(K)$ with $p=\frac{1}{2}(n-m)$ is the probability that an equation in K has exactly m real roots. Other probabilities found are those for an equation in K to have (1) an assigned number h of roots, of which m are real; (2) h assigned real and complex numbers as roots. *M. Marden* (Milwaukee, Wis.).

Seitz, J., and Winkelbauer, K. Remark concerning a paper of Kolmogorov and Prochorov. Čehoslovack. Mat. Ž. 3(78), 89–91 (1953). (Russian. English summary)

The authors claim to give counter-examples to several theorems of Kolmogorov and Prohorov [Uspehi Matem. Nauk (N.S.) 4, no. 4(32), 168–172 (1949); these Rev. 11, 119], and then to formulate these theorems correctly. The authors overlook the fact that condition w of the paper

cited already precludes stopping rules which depend upon observations subsequent to the one at which stopping occurs; the authors' counter-examples actually employ such rules. *J. Wolfowitz* (Ithaca, N. Y.).

Tumanyan, S. H. On the asymptotic distribution of the χ^2 criterion. Doklady Akad. Nauk SSSR (N.S.) **94**, 1011–1012 (1954). (Russian)

Let $p_0, p_1, \cdots, p_s$ be the probabilities for $s+1$ possible outcomes of an experiment. Let $m_0/n, m_1/n, \cdots, m_s/n$ be the relative frequencies of these outcomes in n independent trials. Let

$$\chi^2=\sum_{i=0}^{s}\frac{n}{p_i}\left(\frac{m_i}{n}-p_i\right)^2, \quad F(x)=\Pr\{\chi^2<x\}$$

and

$$K_s(x)=\frac{1}{2^{s/2}\Gamma(s/2)}\int x^{(s-1)/2}e^{-x/2}dx \text{ for } x\geqq 0, \ =0 \text{ for } x\leqq 0.$$

The author states without proofs the following two results. Let n, s and p_i vary simultaneously in such a way that $\min_{0\leq i\leq s} np_i\to\infty$. Then for all x, $F(x)\to K_s(x)$. Let n, s, k, and p_i vary simultaneously in such a way that $s\to\infty$, $k/s\to 0$, and the product np_i for $0\leqq i\leqq k$ satisfies the inequality $c\leqq np_i\leqq C$, for $c>0$, and C fixed, and for $i>k$ satisfies the condition, $\min_{k\leq i\leq s} np_i\to\infty$. Then, for all u,

$$F(s+u\sqrt{2s})\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u}e^{-u^2/2}du.$$

J. L. Snell (Hanover, N. H.).

Parzen, Emanuel. On uniform convergence of families of sequences of random variables. Univ. California Publ. Statist. **2**, 23–53 (1954).

$\theta\in\Omega$ étant un paramètre de nature quelconque, à chaque valeur de θ on associe une suite $\{X_j\}$ de variables aléatoires mutuellement indépendantes; l'auteur établit des critères assurant que $S_n=X_1+\cdots+X_n$ obéit à la loi des grands nombres (en probabilité ou presque-sure), on a la tendance vers la loi normale, uniformément par rapport à θ; il examine en particulier le cas où les X_j ont la même fonction de répartition. Pour cela, considérant des fonctions de répartition $F^\theta(x)$, $F_n^\theta(x)$ $(n=1, 2, 3, \cdots)$ dépendant de θ, il établit d'abord des critères portant sur les caractéristiques de $F^\theta(x)$, $F_n^\theta(x)$ et assurant la convergence uniforme de $F_n(x)$ vers $F(x)$ lorsque $n\to+\infty$. De ses résultats l'auteur déduit l'uniformité de certaines propriétés asymptotiques de tests ou estimations statistiques, et obtient en particulier un théorème de Glivenko-Cantelli uniforme. Il n'est pas possible de réproduire ici ses énoncés. *R. Fortet* (Paris).

Gnedenko, B. V. A local limit theorem for densities. Doklady Akad. Nauk SSSR (N.S.) **95**, 5–7 (1954). (Russian)

Let $\xi_1, \xi_2, \cdots$ be random variables with a common distribution function. Let $s_n=\sum_1^n(\xi_j/B_n)-A_n$, and let p_n be the distribution density of s_n. Let p be the distribution density of some stable distribution. Then if (1) the distribution of ξ_j is in the domain of attraction of the p distribution, and if (2) for some m the distribution of $\sum_1^m\xi_j$ satisfies a Lipschitz condition, it is proved that p_n exists, for large n, that A_n, B_n can be chosen to make s_n have asymptotically the p distribution, and, with this choice, $\sup_x|p_n(x)-p(x)|\to 0$. A trivial converse is easily formulated. If conditions (1) and (2) are satisfied, and if ξ_j has a finite moment of order $k\geqq 3$, an expansion of p_n is given with an error term of order $o(n^{-(k-2)/2})$. *J. L. Doob* (Urbana, Ill.).

Pinsker, M. S., and Yaglom, A. M. On linear extrapolation of random processes with stationary nth increments. Doklady Akad. Nauk SSSR (N.S.) **94**, 385–388 (1954). (Russian)

The authors extend the usual criteria and classifications of linear least squares extrapolation (prediction) theory, continuous parameter case, from stationary (wide sense) stochastic processes to stochastic processes having stationary (wide sense) nth order increments. These processes were defined and discussed in a previous paper [same Doklady (N.S.) **90**, 731–734 (1953); these Rev. **15**, 238]. Proofs are omitted. *J. L. Doob* (Urbana, Ill.).

Zolotarëv, V. M. On a problem from the theory of branching random processes. Uspehi Matem. Nauk (N.S.) **9**, no. 2(60), 147–156 (1954). (Russian)

The author considers the totality of particles whose number changes stochastically in accordance with the following law: In the time interval $(t, t+dt)$ each of the particles, independently of the others, falls apart into k particles, $k=2, 3, \cdots$, with probability $p_kdt+o(dt)$, disappears with probability $p_0dt+o(dt)$, and remains unchanged with probability $1+p_1dt+o(dt)$. We have $p_k\geqq 0$ for $k\neq 1$, $p_1\leqq 0$, and $\sum_0^\infty p_k=0$. Let $f(z)=\sum_0^\infty p_kz^k$, and S be the smallest nonnegative root of $f(z)=0$. Let η be the largest number of particles ever attained in a system which begins with k_0 particles. The author proves: Theorem I:

$$P\{\eta<n+1\}=u_{n-k_0}/S^{k_0}u_n,$$

where $\sum_0^\infty u_nz^n=p_0/f(z)$ for $|z|<S$. Theorem II: If $f'(S)\neq 0$, then $E\eta$ is finite, and always $k_0\leqq E\eta\leqq -k_0p_0/Sf'(S)$. If $f'(S)=0$, then $E\eta=\infty$. Theorem III: If $f'(S)\neq 0$, if there is a positive root of $f(z)=0$ not equal to 1, and if the third derivative M_3 of $f(z)$ at the largest positive root R of $f(z)=0$ is finite, then

$$P\{\eta\geqq n+1\}=\frac{-Sf'(S)}{Rf'(R)}\left[\left(\frac{R}{S}\right)^{k_0}-1\right]\left(\frac{S}{R}\right)^n+o\left(\frac{S^n}{nR^n}\right).$$

If $f'(S)=0$ and $M_3<\infty$, then

$$P\{\eta\geqq n+1\}=\frac{k_0}{n}+o\left(\frac{1}{n}\right).$$

J. Wolfowitz (Ithaca, N. Y.).

Lukacs, Eugene. On strongly continuous stochastic processes. Sankhyā **13**, 219–228 (1954).

L'auteur reprend la définition des fonctions aléatoires $Y(t)$ fortement continues donnée par H. B. Mann [Nat. Bur. Standards Appl. Math. Ser. no. **24** (1953); ces Rev. **14**, 663]: étant donné un intervalle fermé (a, b), soient ϵ, δ deux nombres >0 quelconques, S un ensemble fini quelconque de points de (a, b), $\mathcal{E}(\epsilon, \delta, S)$ l'événement que $|Y(t_j)-Y(t_k)|<\epsilon$ pour tous les couples (t_j, t_k) de points de S tels que $|t_k-t_j|<\delta$; $Y(t)$ est fortement continue sur (a, b) si pour tout couple de nombres >0, ϵ et η, il existe un nombre >0 tel que $\Pr[\mathcal{E}(\epsilon, \delta, S)]>1-\eta$ quelque soit S; cette propriété dépend uniquement de la loi temporelle de $Y(t)$; l'auteur démontre que si $Y(t)$ est à accroissements indépendants et fortement continue sur (a, b), $Y(b)-Y(a)$ est une variable aléatoire Laplacienne, résultat analogue à un théorème connu de P. Levy; il en déduit diverses conséquences et applications, en particulier au processus de Wiener-Levy. *R. Fortet* (Paris).

Bochner, S. Boundedness and stationarity of time series. Proc. Nat. Acad. Sci. U. S. A. 40, 289–294 (1954).

The author considers what are sometimes called stochastic processes of second order, using Hilbert-space language. For each real number t, $x(t)$ is an element of a fixed Hilbert space, with $\int_{-\infty}^{\infty}\|x(t)\|^2(1+t^2)^{-1}dt<\infty$. Then $x(t)$ has a Fourier representation $\int_{-\infty}^{\infty}e^{i\alpha t}dE(\alpha)$, where $E(\alpha)$ is uniquely determined, aside from normalizations. The main emphasis is on the solution of equations of the form $\sum c_j x(t+w_j)=y(t)$, where $x(t)$ and $y(t)$ are of the above type, and $y(t)$ is specified. The equation is solved completely in important cases. The equation is then generalized by replacing the point functions involved by interval functions. *J. L. Doob.*

Ramakrishnan, Alladi, and Mathews, P. M. Studies on the stochastic problem of electron-photon cascades. Progress Theoret. Physics 11, 95–117 (1954).

This contains a review of methods which have been used by various authors. Calculations are made of the second moment of the number of electrons below a given energy level at a given time, based on the analytical work of Bhabha and Ramakrishnan [Proc. Indian Acad. Sci. Sect. A. 32, 141–153 (1950); these Rev. 14, 62]. *T. E. Harris.*

Mathematical Statistics

Benard, A., and Bos-Levenbach, E. C. The plotting of observations on probability paper. Statistica, Rijswijk 7, 163–173 (1953). (Dutch. English summary)

Rosenbaum, S. Tables for a nonparametric test of location. Ann. Math. Statistics 25, 146–150 (1954).

Let s be the number of values of a random sample of size m from a continuous population π that are greater (smaller) than the largest (smallest) value of a random sample of size n from π. Tables of critical 5% and 1% values of s are given for $m=1(1)50$ and $n=1(1)50$. *D. M. Sandelius.*

Babinin, B. V. A nomogram of the basic statistical distributions and its application to certain problems of sampling. Akad. Nauk SSSR. Inženernyĭ Sbornik 11, 169–180 (1952). (Russian)

The nomogram consists of two Z-charts (superimposed along their diagonals) for the relation (1): $t/\sqrt{n}=D/\sigma$; a second calibration with the σ-scale for $\sigma=\sqrt{[p(1-p)]}$; a second calibration with the t-scale for P, the Gaussian distribution; and a network of n- and t-curves arranged so that by orthogonal projection from a point in it to the P-scale the latter is also a binary scale for the Student distribution. The auxiliary function used to give this binary-scale representation was chosen so that solution of (1) can be accomplished if t is the Student distribution $t(P, n)$. Four problems are formulated and solved with and without using the chart. The reduction in size and lack of a movable t- and P-scale make the reproduced version rather unsatisfactory. No explanation of construction is given nor source of data giving the t-curves for $0<n-1<1$ or n-curves for large n. *R. Church* (Monterey, Calif.).

Dawson, Reed B., Jr. A simplified expression for the variance of the χ^2-function on a contingency table. Biometrika 41, 280 (1954).

Jambunathan, M. V. Some properties of Beta and Gamma distributions. Ann. Math. Statistics 25, 401–405 (1954).

Various theorems are proved concerning the distribution of the product of two Beta variates with specialized parameters. These theorems are used to deduce corresponding theorems for the Pearson Type VI distribution. The results aid in deriving the distribution of the Studentised D^2-statistic under the null hypothesis in two different ways.

L. A. Aroian (Culver City, Calif.).

Gil Pelaez, J. Absolute functions in statistics. Trabajos Estadistica 3, 315–339 (1952); 4, 35–54 (1953). (Spanish. English summary)

The author defines a polygonal function to be a function of the form $y=\sum a_i|X-X_i|+bX+c$. He discusses the differential and integral calculus of functions involving polygonal functions and then discusses some application of polygonal functions to probability and statistics in the section on probability. The use of polygonal functions merely makes it possible in some cases to "replace" the Stieltjes integral by a Riemann integral. Some applications are made to the problem of estimation and to the rank correlation coefficient. Also the author introduces a method of fitting which he calls the method of "quadratic marginals". This is applied to fitting an absolute polynomial of the second degree to the normal distribution which the author calls the "practical normal function". To the reviewer this does not seem to have very many applications.

H. Rubin (Stanford, Calif.).

Barrow, D. F., and Cohen, A. C., Jr. On some functions involving Mill's ratio. Ann. Math. Statistics 25, 405–408 (1954).

In this note we prove that, for all (finite) values of h, $\psi(h)=m_2/m_1^2=[1-h(Z-h)]/(Z-h)^2$ is monotonic increasing, that $2m_1^2-m_2>0$, and that $1<\psi(h)<2$, where Z is the reciprocal of Mill's ratio,

$$Z(h)=e^{-h^2/2}\Big/\int_h^\infty e^{-t^2/2}dt,$$

and where m_1 and m_2 are respectively the first and second moments of a singly truncated normal distribution about the point of truncation. (From the author's summary.)

L. A. Aroian (Culver City, Calif.).

Kruskal, William. The monotonicity of the ratio of two noncentral t density functions. Ann. Math. Statistics 25, 162–165 (1954).

The author gives an elementary proof that the ratio of two noncentral t density functions with the same number of degrees of freedom and with noncentral parameters $\delta_N>\delta_D$ for the numerator and denominator respectively, is strictly increasing in t. Various applications of this result as well as references to proofs for the special case $\delta_D=0$ are given in the note. *M. Sobel* (Allentown, Pa.).

Winkelbauer, Karel. Moments for cumulative sums of random variables. Čehoslovack. Mat. Ž. 3(78), 93–108 (1953). (Russian. English summary)

Some improvements of results of the reviewer [Ann. Math. Statistics 18, 215–230 (1947), Section 7; these Rev. 9, 49] on Wald's equation in sequential analysis.

J. Wolfowitz (Ithaca, N. Y.).

Darmois, G. Analyse générale des liaisons stochastiques. Etude particulière de l'analyse factorielle linéaire. Rev. Inst. Internat. Statistique **21**, 2–8 (1953).

The main problem considered is to deduce the normality of chance variables $\xi_i (i=1, 2, \cdots, k)$ from the hypotheses (i) the ξ_i are independent and (ii) there exist two functions of the ξ_i, $X=f(\xi_1, \xi_2, \cdots, \xi_k)$ and $Y=g(\xi_1, \xi_2, \cdots, \xi_k)$, which are stochastically independent. Special attention is given to the case in which f and g are linear in ξ_i and $k=2$. For this case, the results of S. Bernstein and M. Fréchet are extended by removing certain assumptions they imposed. This proof is generalized to the linear case of n chance variables where $k=n$, i.e., $X_i=f_i(\xi_1, \xi_2, \cdots, \xi_n)$ $(i=1, 2, \cdots, n)$. In another generalization where $n=2$ is fixed the author removes some restrictions imposed by D. Basu in the linear case with two functions and general $k>2$. Another problem deals with the uniqueness of a linear analysis. The method of proof in these problems involving functional equations is of some interest in its own right. Some results are also given for the problem (from Spearman's theory) of deducing the stochastic dependence of X and Y from the hypotheses $X=f(\xi_1, \xi_2)$, $Y=g(\xi_1, \xi_2)$, the ξ_i are independent and the functions are not necessarily linear. These problems have application in factor analysis.

M. Sobel (Allentown, Pa.).

Ghosh, M. N. Asymptotic distribution of serial statistics and applications to problems of nonparametric tests of hypotheses. Ann. Math. Statistics **25**, 218–251 (1954).

A serial statistic $S=S(x_1, \cdots, x_n)$ is defined as a statistic of the form $S=n^{-1}\sum_{i=1}^{n} f_i(x_i, \cdots, x_{i+k-1})$. It is shown that under permutations of the observations the distribution of S converges stochastically to the normal distribution provided the moments of the functions $f_i(x_1, \cdots, x_k)$ satisfy certain general conditions. Similarly, the joint distribution of several serial statistics is shown to converge stochastically to the multivariate normal distribution. The asymptotic power of tests of randomness based on serial statistics is investigated if under the alternative hypothesis $x_1, \cdots, x_n$ follow a Markov process. *G. E. Noether.*

Korolyuk, V. S., and Yaroševs'kiĭ, B. I. Study of the maximum discrepancy of two empirical distributions. Dopovidi Akad. Nauk Ukrain. RSR **1951**, 243–247 (1951). (Ukrainian. Russian summary)

The authors treat the same problem as Gnedenko and Korolyuk [Doklady Akad. Nauk SSSR (N.S.) **80**, 525–528 (1951); these Rev. **13**, 570]. Since the latter paper is not cited, the present paper is presumably an earlier one.

J. Wolfowitz (Ithaca, N. Y.).

Fortet, R., et Mourier, E. Convergence de la répartition empirique vers la répartition théorique. Ann. Sci. Ecole Norm. Sup. (3) **70**, 267–285 (1953).

Let $Y=\{y\}$ be a separable metric space of points y, and let $B=\{e\}$ be a Borel field of subsets e of Y such that the spheres of Y are in B. Define $\varphi(e, y)=1$ if y is in e, $\varphi(e, y)=0$ if y is not in e. Let $\{X_j\}$ be a sequence of independent chance variables with values in Y and a common probability measure $P(e)$. The chance set function $P_n(e)-P(e)=n^{-1}\sum_{j=1}^{n}[(\varphi(e, X_j)-P(e)]$ is an element of a certain Banach space whose construction by the authors is not easily described here; the norm of $P_n(e)-P(e)$ in this space will be written as $\|P_n(e)-P(e)\|$. Theorem 1: With probability one, $\|P_n(e)-P(e)\|\to 0$. Theorem 2: $E[\|P_n(e)-P(e)\|]\to 0$. These theorems are followed by similar results involving a different norm for $[P_n(e)-P(e)]$, and results on the convergence of means of certain chance functionals on Y. The proofs utilize results of the second author [Thesis, Paris, 1952].

Let $\{(X_{i1}, X_{i2})\}$ be an infinite sequence of independently and identically distributed pairs of real chance variables, let $p=(p_1, p_2)$ be a couple of real parameters, let $F(x|p)$ be the distribution function (d.f.) of $p_1X_{i1}+p_2X_{i2}$, and let $F_n^*(x|p)$ be the empiric d.f. of $(p_1X_{11}+p_2X_{12}), \cdots, (p_1X_{n1}+p_2X_{n2})$. The reviewer announced earlier and then published in the paper reviewed below (A) a proof of

$$(1)\qquad \text{Prob}\,\{\limsup_{n\to\infty}\sup_{p}\sup_{x}|F(x|p)-F_n^*(x|p)|=0\}=1$$

under the restriction (2): X_{i1} and X_{i2} are independently distributed. The authors describe (in French) the restriction (2) as "inutile", by which they apparently mean that in proving (1) they replace restriction (2) by their own restriction (3): The distribution of (X_{i1}, X_{i2}) is absolutely continuous, or by other restrictions which they do not specify. This is their Theorem 5. [Reviewer's note: By a slight modification of the proof in A it can be shown that all restrictions are unnecessary, i.e., (1) holds without any further conditions.] *J. Wolfowitz* (Ithaca, N. Y.).

Wolfowitz, J. Generalization of the theorem of Glivenko-Cantelli. Ann. Math. Statistics **25**, 131–138 (1954).

Let X_i, $i=1, 2, \cdots$ be a sequence of independent vector-valued random variables. Then for every dual vector q, qX_i is a sequence of independent real random-valued variables, and by the theorem of Glivenko-Cantelli, the empirical distribution of the qX_i converges uniformly to the population distribution. In this paper this is proved to be true uniformly in q. The method of proof is to first prove it for the case of a continuous distribution function by reducing the problem to a bounded set of q's and using continuity. In the discrete case the proof is made by essentially eliminating sufficiently large jumps and treating the two parts separately. *H. Rubin* (Stanford, Calif.).

Wolfowitz, J. Estimation by the minimum distance method in nonparametric stochastic difference equations. Ann. Math. Statistics **25**, 203–217 (1954).

The minimum-distance method was introduced by the author previously [Skand. Aktuarietidskr. **35**, 132–151 (1953); these Rev. **14**, 776]. New results on estimation for stochastic difference equations are given. The following is illustrative. Suppose $x_i=\gamma x_{i-1}+u_i$, $-1<\gamma<1$, $i=1, 2, \cdots$. The x_i are stationary; the u_i are assumed only to be independent and identically distributed, in such a manner that $\sum_{i=1}^{\infty}\gamma^i u_i$ converges. It is required to estimate γ from observations on $x_1, x_2, \cdots, x_{2n+1}$. Let $q_i(g)=x_i-gx_{i-1}$. Let $A_n(x|g)$ be the empirical (cumulative) distribution function of $q_2, \cdots, q_{2n+1}$, $C_n(x, y|g)$ the empiric distribution of the pairs $(q_2, q_3; q_4, q_5; \cdots; q_{2n}, q_{2n+1})$, $B_n(x, y|g)=A_n(x|g)A_n(y|g)$. Let g_n be any Borel-measurable function of $x_1, \cdots, x_{2n+1}$ such that $|g_n|<1$ and

$$\delta[B_n(x, y|g_n), C_n(x, y|g_n)] < \frac{1}{n}+\inf_{g}\delta[B_n(x, y|g), C_n(x, y|g)],$$

where the δ-distance is the supremum of the absolute difference. Then g_n is a "superconsistent" estimate of γ; that is, g_n converges to γ with probability 1. Use is made of a generalization due to the author [see the paper reviewed

above] of the theorem of Glivenko-Cantelli. Other problems considered: (A) Estimate α when $x_i = u_i + \alpha u_{i-1}$; (B) estimate β when $y_i = \beta y_{i-1} + u_i$, $x_i = y_i + v_i$. In each case x_i is the observable; $|\beta|, |\gamma| < 1$; the u_i satisfy conditions similar to those given above; the v_i are independent of one another and have a common distribution but are not necessarily independent of the u_i. *T. E. Harris.*

Schmetterer, L. Zum Sequentialverfahren von Robbins und Monro. Monatsh. Math. 58, 33–37 (1954).

The author gives a bound, on the second moment about θ, of the nth approximation to a quantile θ of an unknown regression function, in the approximating procedure of Robbins and Monro [Ann. Math. Statistics 22, 400–407 (1951); these Rev. 13, 144]. The bound is derived under conditions somewhat more general than those used in the paper cited to prove convergence in mean square, and approaches zero as $n \to \infty$. *J. Wolfowitz* (Ithaca, N. Y.).

Chartier, F. L'estimation statistique dans le cas d'observations non indépendantes. Etude d'un cas particulier. Publ. Inst. Statist. Univ. Paris 1, no. 4, 3–39 (3 plates) (1952).

This paper is in large part expository, the general purpose being to point out that in many cases the assumption of the independence of the observations is perhaps unjustified, and to discuss methods of testing independence and estimating parameters in the case of non-independent observations. Attention is mostly confined to the case where the conditional distribution of X_i, given the values of $X_1, \cdots, X_{i-1}$, depends only on the value of X_{i-1} (and, of course, not even on that in the case of independence), and to the particular case where the distribution of X_1 is normal, with mean θ and standard deviation σ, while the conditional distribution of X_i, given X_{i-1}, is normal with mean $\theta + r(X_{i-1} - \theta)$ and standard deviation $\sigma\sqrt{(1-r^2)}$.

First, the author describes two distribution-free tests of the hypothesis of independence of successive observations. These seem to be in error, and will be more fully described in the last paragraph of this review. Next, the author discusses the use of the serial correlation coefficient in testing independence, and its distribution in the normal case. Finally, maximum likelihood estimates of the parameters θ, r and σ defined above, and the distribution of these estimates, are discussed.

The distribution-free tests of independence are described (p. 9) as follows. Denote the sample median of the observations $X_1, \cdots, X_N$ by M', and define $d_i = X_i - M'$. In the sequence $d_1, \cdots, d_N$ replace a negative value by the symbol n, a positive value by the symbol p. The first test involves the lengths of the subsequences ending in the same symbol as the whole sequence. Thus, if the sequence is $pnnppnnp$, the subsequences are p, nnp, p, nnp, of lengths 1, 3, 1, 3. The second test involves the lengths of subsequences of identical symbols, these lengths being 1, 2, 2, 2, 1 in our example. In either test, the test is made by comparing the observed lengths with the lengths expected when the variables $X_1, \cdots, X_N$ are independently and identically distributed. However, for each test the author states that the observed lengths can vary between 1 and N inclusive, which is incorrect. From his discussion, it appears that he has confused the sample median M' with a common population median M, and hence has forgotten that half of the values d_i will be negative, the other half positive. *L. Weiss.*

Good, I. J. The population frequencies of species and the estimation of population parameters. Biometrika 40, 237–264 (1953).

Suppose that a random sample of size N is drawn from a multinomial population, where the number of classes is not necessarily known. Then consider the problem of estimating the population frequency of each class and in particular the total population frequency of alternatives in the sample. Starting from the basic formula

$$\mathcal{E}(q_r^m) = \frac{(r+m)^{(m)} \mathcal{E}_{N+m}(n_{r+m})}{(N+m)^{(m)} \mathcal{E}_N(n_r)}$$

we obtain an estimate of the population frequency of a class in which r individuals are observed as the ratio of the expected number of classes of $r+1$ individuals to the expected number of classes of r individuals. The author gives a smoothing method of estimating that expectation and hence obtaining an estimate of the population frequency. He also discusses the estimation problem on the assumption that the frequencies are known but it is not known which class has which frequency, or, essentially, if the a priori distribution of the probability of a class to which an individual drawn at random belongs is known. The author fails to observe that it is unnecessary to assume that the number of classes is finite. *H. Rubin* (Stanford, Calif.).

Sarhan, A. E. Estimation of the mean and standard deviation by order statistics. Ann. Math. Statistics 25, 317–328 (1954).

Let $y_1, y_2, \cdots, y_n$ be a random sample of size n drawn from a continuous p.d.f. and let $y_{(1)} < y_{(2)} < \cdots < y_{(n)}$ be the observations rearranged in order of size. The author finds explicitly "ordered least-square estimates"

$$\left(\sum_{i=1}^{n} \alpha_{1i} y_{(i)}, \quad \sum_{i=1}^{n} \alpha_{2i} y_{(i)} \right)$$

of the means and standard deviations of the two-parameter rectangular, triangular, exponential, and Laplace p.d.f.'s. The following tables, all for $2 \leqq n \leqq 5$, are given. Tables 1 and 2: $E(X_r)$ and cov (X_r, X_s), $1 \leqq r, s \leqq n$, for the standardized triangular and Laplace p.d.f.'s. Table 3: α_{1i} and α_{2i}, $1 \leqq i \leqq n$, for the rectangular, triangular, normal, and Laplace p.d.f.'s. Table 4: efficiencies of sample mean, midrange, and median (when compared with the "best linear" estimate) for the same p.d.f.'s as Table 3. *B. Epstein.*

Maniya, G. M. Practical application of the estimate of the maximum of bilateral deviations of an empirical distribution curve in a given interval of growth of a theoretical law. Soobščeniya Akad. Nauk Gruzin. SSR 14, 521–524 (1953). (Russian)

The author gives a table for the case $\theta_1 + \theta_2 = 1$ of the asymptotic distribution of a chance variable described in A: G. M. Maniya, Doklady Akad. Nauk SSSR (N.S.) 69, 495–497 (1949) [these Rev. 11, 261], where the result is stated without proof. Anderson and Darling [B: Ann. Math. Statistics 23, 193–212 (1952); these Rev. 14, 298] state that the formula in A is incorrect and give another formula. In the present paper the author uses the same formula as in B but makes no mention of any error in A. The author refers to a paper in the Učenye Zapiski Moskov. Gos. Pedagog. Inst. im. Potemkin. 16 (1951). Presumably the result is proved there, but this paper is unavailable to the reviewer. *J. Wolfowitz* (Ithaca, N. Y.).

Graybill, Franklin A. On quadratic estimates of variance components. Ann. Math. Statistics **25**, 367–372 (1954).

For the linear model in the analysis of variance for the two-fold classification $Y_{ijk}=\mu+a_i+b_{ij}+e_{ijk}$, in which the a_i, b_{ij} and e_{ijk} are independent random variables with finite variances, the author shows that if a_i, b_{ij}, e_{ijk} possess finite fourth moments, the usual estimate of the variance of the b_{ij} obtained by the analysis of variance is the quadratic unbiased estimate with the minimum variance. The method of proof extends to the k-fold classification for which he states the corresponding theorem. *C. C. Craig.*

Downton, F. Least-squares estimates using ordered observations. Ann. Math. Statistics **25**, 303–316 (1954).

Consider two-parameter p.d.f.'s of the form $f\{(x-\mu)/\sigma\}/\sigma$. The object of the paper is to estimate the parameters μ and σ by applying the method of least squares to the observations, after they have been arranged in order of size. Such estimates are called "ordered least-square estimates" and have the property of being unbiased and of minimum variance in the class of all unbiased estimates which are linear in the ordered observations. A general theory of such estimates has been given by Lloyd [Biometrika **39**, 88–95 (1952); these Rev. **14**, 65]. In this paper explicit estimates are found for a class of two-parameter p.d.f.'s of the form given in the first sentence. This class contains the rectangular and right-triangular distributions as special cases and the exponential distribution as a limiting case. In the last part of the paper, the author gives a general property of ordered least-squared estimates of the parameter λ in p.d.f.'s having the form $f(x/\lambda)/\lambda$. The result is that the ordered least-square estimate of the scale parameter λ in a Pearson Type III distribution is identical with the maximum-likelihood estimate. *B. Epstein* (Detroit, Mich.).

Hamaker, H. C. The efficiency of sequential sampling for attributes. II. Practical applications. Philips Research Rep. **8**, 427–433 (1953).

[For part I see same Rep. **8**, 35–46 (1953); these Rev. **14**, 996.] For sequential sampling of attributes with stopping lines $sn\pm k$, a slide rule for obtaining the OC curve is constructed, and graphs and tables are given which relate s and k to the AOQL, AQL, etc., for certain specifications on these levels. *J. Kiefer* (Ithaca, N. Y.).

Chapman, Douglas G. The estimation of biological populations. Ann. Math. Statistics **25**, 1–15 (1954).

This is a survey of statistical models of the methods used or usable in the estimation of sizes or other parameters of stationary or non-stationary animal populations. Estimates (mainly for large populations) and their variances and covariances are given. The models include direct and inverse tag-and-sample methods [cf., for the stationary case, Chapman, Biometrics **8**, 286–306 (1952); these Rev. **14**, 777; Goodman, Ann. Math. Statistics **24**, 56–69; these Rev. **14**, 776], a method based on an initial sampling process followed by a selective removal of individuals from the population and a second sampling process, and methods where use is made of data on the effort expended on sampling. Among the new contributions by the author is a tag-and-sample model where a regression assumption is substituted for the usual assumption of strictly multinomial (or Poisson) variation. *D. M. Sandelius* (Göteborg).

Masuyama, Motosaburo. Mathematical note on area sampling. Sankhyā **13**, 241–242 (1954).

The author draws attention to certain known formulae in integral geometry that he has found useful in statistical applications in a previous paper [Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **2**, no. 4, 113–119 (1953); these Rev. **15**, 332]. *H. P. Mulholland* (Birmingham).

Choudhury, P. Sur un test d'indépendance des moyennes et des écarts types d'échantillons extraits d'une population normale. Publ. Inst. Statist. Univ. Paris **1**, no. 2, 41–43 (1952).

This paper, which has a misleading title, considers a test for normality of a population based on sample means and standard deviations $\bar{x}_r$, s_r $(r=1, 2, \cdots, k)$ from k independent samples, the rth sample having n_r observations. It is not clear why the k samples are not pooled to form one sample. The statistic $z=\sum_{r=1}^k n_r\bar{x}_r s_r/\sigma^2$ is proposed; under the hypothesis of normality its density is shown to be a standard Bessel function. The key step involves a Fourier inversion for which the result is stated without any reference; the resulting density is given as a function of z although it appears to be the density of $z'=z/2\pi$. No reason is given for considering z, it is not stated which values of z should be considered as critical, the availability of a table to carry out the test is not discussed. Nothing is said about consistency or about the power of the test against any alternative. *M. Sobel* (Allentown, Pa.).

Epstein, B., and Sobel, M. Some theorems relevant to life testing from an exponential distribution. Ann. Math. Statistics **25**, 373–381 (1954).

A life test on N items is considered in which the common underlying distribution of the length of life of a single item is given by the density $p(x;\theta,A)=\theta^{-1}e^{-(x-A)/\theta}$, $x\geqq A$, and zero otherwise, $A\geqq 0$, $\theta>0$. The problem is to estimate θ when the N items are divided into k sets S_j, each containing $n_j>0$ items, $N=\sum_{j=1}^k n_j$, and each set S_j is observed until the first r_j failures occur, $0<r_j\leqq n_j$. Three different cases are considered: (1) the n_j items in each set S_j have a common known A_j, (2) all N items have a common unknown A, (3) the n_j items in each set S_j have a common unknown A_j. The case $k=1$, $A=0$ was treated by the authors previously [J. Amer. Statist. Assoc. **48**, 486–502 (1953); these Rev. **15**, 143]. A uniformly minimum-variance unbiased estimate θ_i^* of θ is given based on a sufficient statistic. If $R=\sum_{j=1}^k r_j$ is the total number of failures observed, the authors show that $2R\hat{\theta}_i/\theta$ is distributed as χ^2 with $2R$, $2(R-1)$, and $2(R-k)$ degrees of freedom in cases (1), (2), and (3) respectively where $\hat{\theta}_i$ is the maximum likelihood estimate of θ. *L. A. Aroian* (Culver City, Calif.).

Miyasawa, Kôichi. Most stringent tests and invariant tests. Mem. Fac. Sci. Kyūsyū Univ. A. **8**, 57–88 (1953).

The topics mentioned in the title are treated in a manner very similar to that of Hunt and Stein and Lehmann [Lehmann, Ann. Math. Statistics **21**, 1–26 (1950); these Rev. **11**, 528; also Lehmann's Theory of testing hypotheses (mimeographed), Berkeley, 1949]. The main difference is that, in the treatment of invariance, the criterion is stated as constancy of the power function on certain surfaces; however, the author then imposes a restriction on the test which necessitates its invariance in the sense of the above authors. The results obtained by the author in applications to the normal distribution are thus the same as in the cited references. *J. Kiefer* (Ithaca, N. Y.).

Basu, D. Choosing between two simple hypotheses and the criterion of consistency. Proc. Nat. Inst. Sci. India 19, 841–849 (1953).

For the most part this is an expository article on testing simple hypotheses, including an example where no consistent sequence of tests exists and the well-known result that a consistent sequence always exists in the case of independent and identically distributed observations. *J. Kiefer.*

***Gumbel, Emil J. Statistical theory of extreme values and some practical applications. A series of lectures.** National Bureau of Standards Applied Mathematics Series No. 33. U. S. Government Printing Office, Washington, D. C., 1954. viii+51 pp.

Let $\Phi(x)$ be the asymptotic c.d.f. of the largest extreme of a random sample of size n from a population with c.d.f. $F(x)$ and p.d.f. $F'(x)$, and let u and α be defined by $n[1-F(u)]=1$ and $\alpha=nF'(u)$. After an exposition of different types of $F(x)$ and corresponding $\Phi(x)$ there follows a detailed treatment of the case

$$(*)\quad \Phi(x)=\exp\{-\exp[-\alpha(x-u)]\},\quad -\infty<x<\infty.$$

To estimate u and $1/\alpha$ consistently, given a random sample, of size N, from a population with c.d.f. (*), the author suggests (i) solving the equation system $\bar{x}_N=u+(\gamma/\alpha)$, $s_N=\pi/(6^{1/2}\alpha)$, where γ is Euler's constant, the right members are the mean and S.D. of (*), and $\bar{x}_N$ and s_N are the sample mean and S.D., or (ii) taking the geometric means of the vertical and horizontal least squares estimates obtained when fitting, on extreme-value probability paper, the line $x=u+(y/\alpha)$ to N points (x_i, y_i), where x_i is the ith order statistic of the sample and

$$y_i=-\log\{-\log[i/(N+1)]\}.$$

(Cf. the following review.) Various practical applications are treated. *D. M. Sandelius* (Göteborg).

Lieblein, Julius. A new method of analyzing extreme-value data. NACA Tech. Note no. **3053**, 88 pp. (1954).

Let $x_1\leqq x_2\leqq\cdots\leqq x_n$ be the n order statistics of a random sample of size n from a population with c.d.f. $\exp\{-\exp[(x-u)/\beta]\}$. (Here n and β correspond to N and $1/\alpha$ of the preceding review.) The author determines the minimum-variance, unbiased, linear order-statistics estimate $\hat{\xi}_p$ of $\xi_p=u+\beta y_p$, where ξ_p and y_p are the 100P % points of x and the standardized variate y. [For the general case of such order-statistics estimates cf. Lloyd, Biometrika **39**, 88–95 (1952); these Rev. **14**, 65.] Necessary tables for numerical determination of $\hat{\xi}_p$, its variance and the confidence limits of ξ_p are given for various P-values and $2\leqq n\leqq 6$. For $n>6$ division of the original sample into subgroups and combination of subgroup estimates of ξ_p is recommended. Comparing the variance of $\hat{\xi}_p$ with estimates (from artificial samples) of the mean square errors of the ξ_p-estimates obtained by Gumbel's methods (i) and (ii) (cf. the preceding review), the author finds his own method more efficient than (i) for samples of about 20 or more and $P=0.95$ and more, and up to twice as efficient as (ii) for the same range of values of P and for samples of 10 or more. *D. M. Sandelius* (Göteborg).

***Wold, Herman. A study in the analysis of stationary time series. 2d ed. With an appendix by Peter Whittle.** Almqvist and Wiksell, Stockholm, 1954. viii+236 pp. Sw. kr. 28.00.

This second edition is a reprint of the author's thesis [Uppsala, 1938]. Two appendices (Appendix A, On the ω^2-method for testing goodness of fit; Appendix B, On the quantitative significance of correlation coefficients) of the 1938 version were omitted from the present edition and were replaced by two new appendices. Appendix 1 contains notes to the second edition and comments on a few points of the main text. Appendix 2 was written by P. Whittle and is entitled "Some recent contributions to the theory of stationary processes". This appendix gives a very concise survey of the progress of the theory since 1938. It consists of two chapters. The first deals with spectral theory. Cramér's representation of a stationary process by means of a process with uncorrelated increments is discussed. The decomposition of the process corresponding to the Lebesgue decomposition of the spectral function is treated and the deterministic, respectively nondeterministic, character of these components is investigated. Linear operations and their use in prediction theory are also mentioned. The second chapter deals with statistical inference in time series analysis. The relation between observed and residual moments is discussed and the cumulants of linear functions of the autocovariances are calculated. Least-squares estimation of the parameters of the spectral function is treated. Finally the theory of testing hypotheses is considered, first for the purely nondeterministic case and then also for a scheme containing a deterministic component. A few minor changes were made in the original list of references and a separate bibliography for Appendix 2 was added. *E. Lukacs* (Washington, D. C.).

Sargan, J. D. An approximate treatment of the properties of the correlogram and periodogram. J. Roy. Statist. Soc. Ser. B. **15**, 140–152 (1953).

By neglecting end effects from the start, the author obtains some simple derivations of the more important properties of correlograms and periodograms of sequences derived from autoregressive models. He applies the periodogram and several popular statistical tests to an autoregressive model for the Beveridge wheat price index. It is remarkable that failures of the model which are very prominent on the periodogram pass undetected by these tests, with the exception of that of Quenouille. *A. Blake.*

Fourgeaud, Claude, et Féron, Robert. Quelques remarques sur l'estimation des variations saisonnières. Publ. Inst. Statist. Univ. Paris **1**, no. 3, 21–25 (1952).

The author considers a discrete time-series which is the sum of a periodic component, a linear trend and a random component. Least squares estimates of its parameters and the variance of the estimates are given. The grouping of observations and the effect of missing observations is considered. No proofs are given. *E. Lukacs.*

Blanc-Lapierre, A. Sur quelques modèles statistiques suggérés par l'étude de l'effet de scintillation. Publ. Inst. Statist. Univ. Paris **2**, no. 3, 3–17 (1953).

Mathematically, this paper deals with random functions associated with Poisson processes and utilizes earlier work of the author [C. R. Acad. Sci. Paris **221**, 375–377 (1945); these Rev. **7**, 211] and of Fortet [Proc. 2nd Berkeley Symposium Math. Statistics and Probability, 1950, Univ. of California Press, 1951, pp. 373–385; these Rev. **13**, 958]. Model 1: let t_j be points of occurrence of a Poisson process with constant parameter ρ; with each t_j is associated an impulse of unit intensity and random duration l_j; l_j has an exponential distribution, mean λ; different l_j are independent. Let $x(t)$ be the sum of the impulses extant at t. The

spectral density of $x(t)$ is $f(\nu)=2\lambda/(1+4\pi^2\nu^2\lambda^2)$; $x(t)$ is approximately Gaussian when $\lambda\rho$ is large. Physically, it has been observed that the spectral density should be proportional to $1/\nu$ over a wide range of frequencies. This can be approximately attained by proper superposition of processes with different values of λ. Several related models are discussed. *T. E. Harris* (Santa Monica, Calif.).

Samson, Edward W. Fundamental natural concepts of information theory. Communications Laboratory, Electronics Research Division, Air Force Cambridge Research Center, Cambridge, Mass., Rep. **E5079**, 25 pp. (1951).

Samson, Edward W. Theory of information: The basic theorems on system uncertainty. Communications Laboratory, Electronics Research Directorate, Air Force Cambridge Research Center, Cambridge, Mass., Tech. Rep. **53-37**, vii+25 pp. (1953).

Samson, Edward. Information theory: Questions and uncertainties. Communications Laboratory, Electronics Research Directorate, Air Force Cambridge Research Center, Cambridge, Mass., Tech. Rep. **54-1**, viii+45 pp. (1954).

An exposition of some sections of Shannon's information theory, intended to bring out their intuitive and semantic implications. The author regards as his own main contribution the interpretation of $-\log(p)$ as the surprise felt when an event of prior probability p takes place. The last paper includes a formalisation of one's advance attitude to an experiment. *P. Whittle* (Wellington).

Mathematical Economics

Leontief, Wassily. Mathematics in economics. Bull. Amer. Math. Soc. **60**, 215–233 (1954).

Expository paper. *H. S. Houthakker.*

on order

✶Williams, J. D. The compleat strategyst, being a primer on the theory of games of strategy. With pictorial illustrations by Charles Satterfield. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1954. xiii+234 pp. $4.75.

Charnes, A. Constrained games and linear programming. Proc. Nat. Acad. Sci. U. S. A. **39**, 639–641 (1953).

The mixed strategies of a matrix game are vectors with non-negative components and component sum equal to one. A "constrained game" results when the mixed strategies are subjected to further linear inequalities. This note provides a transcription of such a game into a linear program, thus simultaneously proving their solvability and opening an avenue to computation via the simplex method.

H. W. Kuhn (Bryn Mawr, Pa.).

Franckx, Ed. La théorie des corps convexes non séparables et la théorie des jeux. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 18–24 (1954).

This note joins the long list of published proofs of the main theorem of the theory of the zero-sum two-person game. The method used is well known; see, for example, Theorem 6 in the expository article by D. Gale [Activity analysis of production and allocation, Wiley, New York, 1951, pp. 287–297; these Rev. **13**, 60]. *H. W. Kuhn.*

Rosenblatt, Murray. An inventory problem. Econometrica **22**, 244–247 (1954).

The problem treated is a simple grain storage model which falls within the setup studied by Dvoretzky, Kiefer, and Wolfowitz [Econometrica **20**, 187–222 (1952); these Rev. **13**, 856]. Restricting himself to a class of linear storage policies and assuming that the long-run situation approaches stationarity, the author computes the member of this class which minimizes the average risk per time period for a certain quadratic loss function as the number of periods becomes infinite. *J. Kiefer* (Ithaca, N. Y.).

Herstein, I. N., and Milnor, John. An axiomatic approach to measurable utility. Econometrica **21**, 291–297 (1953).

If S is a mixture set (i.e., closed under formal convex combinations) that is completely ordered, the existence of a measurable utility (i.e., a real-valued, order-preserving, linear function) on S is of considerable interest to economists. This paper establishes the necessity (obvious) and sufficiency of the following three axioms relating the order and mixing. 1. The set S is completely ordered by $\geqq$. (The relation $a=b$ is not identity; it means both $a\geqq b$ and $b\geqq a$, and is to be read "a is indifferent to b".) 2. For any $a,b,c\in S$, the sets $\{\alpha\mid\alpha a+(1-\alpha)b\geqq c\}$ and $\{\alpha\mid c\geqq\alpha a+(1-\alpha)b\}$ are closed. 3. If a, $a'\in S$ and $a=a'$, then for any $b\in S$, $\frac{1}{2}a+\frac{1}{2}b=\frac{1}{2}a'+\frac{1}{2}b$. *H. W. Kuhn* (Bryn Mawr, Pa.).

Malécot, G. Sur l'amortissement des fluctuations économiques. (Examen critique de la théorie de Tinbergen.) Publ. Inst. Statist. Univ. Paris **2**, no. 3, 55–65 (1953).

After some introductory remarks about the construction and statistical estimation of macrodynamic models, there is a brief exposition of one such, constructed by Tinbergen for the U. S., 1919–32 [Les fondements mathématiques de la stabilisation du mouvement des affaires, Hermann, Paris, 1938]. *R. Solow* (Cambridge, Mass.).

✶Studies in econometric method. By Cowles Commission research staff members. Edited by Wm. C. Hood and Tjalling C. Koopmans. Cowles Commission Monograph No. 14. John Wiley & Sons, Inc., New York; Chapman & Hall, Limited, London, 1953. xix+323 pp. $5.50.

A study of economic models of the "structural" type. Following up Monograph 10 in the same series [1950; these Rev. **12**, 431] by a less technical account, the report is mainly expository, but brings also new results of continuing research. The emphasis is on the theoretical-statistical analysis, which is impeccable in formal respects, while the applied analysis is less satisfactory and leaves doubt about the rationale of the methods developed.

In the pioneer works of J. Tinbergen [e.g., Rev. Econ. Studies **7**, 73–90 (1940)] the model is always constructed as a causal chain, with one explanatory relation (behaviour relation) for each endogenous variable. Compared with such "recursive" models, the structural approach is more general, for the behaviour relations are here mixed with other types of relationship, usually an equilibrium condition (between demand and supply, say, or between savings and investment) or an identity (say, national income = salaries + profits). The mixed nature of the model, however, is disregarded in the structural approach, all parameters being subjected to simultaneous estimation by a maximum likelihood (m.l.) procedure. At bottom, it is this uniform treatment of disparate relationships that gives rise to certain peculiar features of the methods under review, notably: (1) Being entirely formal, the estimation does not take into

account the subject-matter specification of the model in terms of causality, autonomy, etc., a specification which is essential for guiding the estimation in the recursive approach [cf. Exercise IV, 32 a-c in the reviewer's book, Demand analysis [Wiley, New York, 1953], where equilibria and identities are treated as constraints upon the behaviour relations, in line with classical least-squares methods]; (2) the m.l. estimation turns upon hypothetical properties of linear combinations of the given relations, combinations which are formal constructs of little or no concern for the applications, and their hypothetical specification refers to coefficient patterns and residual intercorrelations about which there is in practice little or no a priori evidence; (3) in particular, the "reduced form" is made up by such combinations, and, for example, in the case when the model has no behaviour relation for one of the endogenous variables, say price p_t, the reduced form relation for p_t, say $R(p)$, cannot be interpreted as a behaviour relation for p_t, but will be largely accidental, raising doubt about $R(p)$ as an adequate instrument for the estimation.

The report contains 10 papers by 11 authors. Three of the introductory papers are reprints, with some revision in conformity with later developments of the structural approach. The main paper is "The estimation of simultaneous linear economic relationships" by T. Koopmans and W. Hood [pp. 112–199]. This is a largely self-contained exposition of the m.l. estimation of structural models. The style is lucid, and the mathematical difficulties are masterfully handled (e.g., a device of step-wise maximation is employed to great didactic advantage). Several variants of specifying assumptions are discussed in detail, showing how the estimation rapidly becomes more complicated with increasing level of ambition in the specification. Special attention is paid to cases when m.l. estimation can be replaced by least-squares estimation [Reviewer's note: Conditions $(A)+(B)$, p. 133, for least-squares regression to give consistent estimates can be relaxed so as to allow residual autocorrelation; see H. Wold, Econometrica **19**, 475–477 (1951)]. In "The computation of maximum-likelihood estimates of linear structural equations" by H. Chernoff and N. Divinsky [pp. 236–302] numerical illustrations with complete computation schemes are given on the basis of a model with 3 behaviour relations and 3 identities. In "Causal ordering and identifiability" by H. A. Simon [pp. 49–74] the structural approach is commented from the viewpoint of an ordering of the variables which formally constitutes a generalization of the causal chain in the recursive approach. The concept of causality introduced by Simon is mathematical-formal, and its relation to the common-sense concept is not clear to the reviewer [cf. also Simon, J. Philos. **49**, 517–528 (1952)]. *H. Wold.*

Mathematical Biology

*Geppert, M. P. **Anwendungen der Mathematik auf Biologie, Medizin und Bevölkerungswissenschaft.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 3. Angewandte Mathematik, Teil I, pp. 205–231. Verlag Chemie, Weinheim, 1953. DM 20.00.

Bennett, J. H. On the theory of random mating. Ann. Eugenics **18**, 311–317 (1954).

For a population of sexually reproducing disomic organisms mating at random in non-overlapping generations, the basic relations satisfied by the gametic frequencies in consecutive generations constitute a system of difference equations. With more than three linked factors, the equations are quadratic, while in cases with fewer linked factors they are linear. Introducing principal components, the author shows that the general problem can be reduced to a linear one whence follows readily the solution in a clear form. In case of possible differences between the sexes, a modification to be made is stated. Further, a problem of random-in-time mating in a population of virus or haploid particles is also discussed. *Y. Komatu* (Tokyo).

Patlak, Clifford S. The effect of the previous generation on the distribution of gene frequencies in populations. Proc. Nat. Acad. Sci. U. S. A. **39**, 1063–1068 (1953).

The gene frequency of a population in S. Wright's theory is subject to a diffusion process, and therefore the whole development depends only on the form of the two arbitrary coefficients. The analytic form of these coefficients in some special cases has been studied by Wright [same Proc. **31**, 382–389 (1945); these Rev. **7**, 319]. Although the present author expresses himself in a different way, from a mathematical point of view the paper contains the remark that the coefficients can also depend on the environment. A discussion is given of how the coefficients might be decomposed into terms which account for the inner biological effects, migration, environment, etc. *W. Feller.*

Tarabini, Vera. Sulle fluttuazioni biologiche. Boll. Un. Mat. Ital. (3) **8**, 422–428 (1953).

L'auteur reprend les équations différentielles de Volterra sur l'evolution de deux espèces dévorée et dévorante (en nombres respectifs N_1 et N_2) étudiées dans l'ouvrage Leçons sur la théorie mathématique de la lutte pour la vie [Gauthier-Villars, Paris, 1931]. Mais il introduit dans la 1ère équation (en dN_1/dt) un terme d'amortissement $\delta_1N_1-\lambda_1N_1^2$ ($\delta_1>0$, $\lambda_1>0$). Alors s'il y a un état d'équilibre stable, N_1 et N_2 tendent vers les valeurs d'équilibre. S'il y a instabilité, les oscillations tendent à devenir périodiques non amorties. Cela est à comparer avec l'étude développée par le reférant [Ann. Mat. Pura Appl. (4) **9**, 57–74 (1931)] lorsqu'on introduit dans la même 1ère équation un terms d'amortissement $-\lambda N_2$ dépendant de la 2ème espèce. *M. Brelot.*

Gulliksen, Harold. A generalization of Thurstone's learning function. Psychometrika **18**, 297–307 (1953).

Thurstone's equation [J. Gen. Psychol. **3**, 469–493 (1930)] giving the probability of a correct response (p) as a function of practice time (t) when punishment and reward have equal effects has been generalized to the case where the effect of punishment is not necessarily equal to the effect of reward. Since the general equation is somewhat unwieldy, three special cases are considered, where reward has no effect, where punishment has no effect, and where these effects are equal. Equations are given together with tables for making a rectified plot for each of the three special cases. (Author's summary.) *D. G. Kendall* (Oxford).

TOPOLOGY

Zarankiewicz, K. The solution of a certain problem on graphs of P. Turan. Bull. Acad. Polon. Sci. Cl. III. 1, 167–168 (1953).

A is a set of p points and B is a set of q points in the Euclidean plane, and every point of A is joined to every point of B by a simple arc lying in the plane and having the point of A and the point of B concerned as its two end points. No three arcs have an interior point (a point that is not an end point) in common. Then there are at least $K(p, q)$ intersection points between the arcs, and the arcs divide the plane into at least $L(p, q)$ regions, where

$$K(2k, 2n)=(k^2-k)(n^2-n),$$
$$K(2k, 2n+1)=(k^2-k)n^2,$$
$$K(2k+1, 2n+1)=k^2n^2,$$
$$L(2k, 2n)=(k^2-k)(n^2-n)+4nk-2(n+k)+2,$$
$$L(2k, 2n+1)=(k^2-k)n^2+4nk-2n+1,$$
$$L(2k+1, 2n+1)=n^2k^2+4nk+1.$$

It is shown by means of an example that the lower bounds $K(p, q)$ and $L(p, q)$ may actually be attained. The proofs of these results will be published in Fundamenta Mathematicae 41. *G. A. Dirac* (London).

Luce, R. Duncan. Networks satisfying minimality conditions. Amer. J. Math. 75, 825–838 (1953).

The paper is concerned with the structure of finite nonreflexive oriented graphs, which are here called networks. A network is connected if there is a chain (oriented path) from every node to every other node, otherwise it is disconnected. Maximal connected subnetworks are called components. A network with every possible link present is called a complete graph. A network has degree 0 if it is disconnected; it has degree k, $k>0$, if it can be made disconnected by removing k suitably chosen links, but not by removing fewer than k links; it is k-minimal if it has degree k but after any one of the links is deleted the remaining network has degree $k-1$; it has degree $(-k)$, $k\geqq 0$, if it can be made connected by adding $k+1$ suitably chosen links but not by adding fewer than $k+1$ links; it is (-0)-minimal if it is disconnected but becomes connected when any new link is added; it is $(-k)$-minimal, $k\geqq 1$, if it has degree $-k$ but when any new link is added the resulting network has degree $-k+1$; it has index κ if it is connected and can be made disconnected by deleting κ suitably chosen nodes, but not by deleting fewer than κ nodes; it is h-transitive if it contains a chain with h links connecting two distinct nodes, and if whenever a chain with more than h links leads from a node a to a second node b, the link (ab) belongs to the network, while whenever a chain with not more than h links leads from c to d, (cd) does not belong to the network; it is uniform if every connected subnetwork has degree 1.

The principal results established are the following. 1. A network is $(-k)$-minimal if and only if either (i) it consists of $k+1$ complete components with no link between any two of them, or (ii) it contains a network X, where X consists of $k+1$ complete components with no link between any two of them, and a complete component Y, and either (a) all possible links of the form (xy), $x\varepsilon X$, $y\varepsilon Y$, and none of the form (ys), or (b) all possible links of the form (yx) and none of the form (xy). 2. If a connected network has m nodes, degree k and index κ, then $\kappa\leqq k\leqq(m-1+\kappa)/2$. 3. If a network has degree k then from any node a to any other node b there is a set of at least k chains such that no two have a common link. 4. If a network with m nodes is connected and h-transitive, $h>1$, then it is uniform or it is a circle (which is 2-minimal and for which $h=m-2$). 5. If a network N is connected, non-minimal and h-transitive, $h>1$, and contains the links (ab) and (ba) then it consists of two connected subnetworks N_a and N_b joined only by (ab) and (ba). One of N_a and N_b is h-transitive and non-minimal, the other is either minimal, or h-transitive and non-minimal. *G. A. Dirac* (London).

Tutte, W. T. A contribution to the theory of chromatic polynomials. Canadian J. Math. 6, 80–91 (1954).

Let G be a finite oriented graph, let $V(G)$ be the set of its vertices, $E(G)$ the set of its edges, and let Q_n be the ring of residue classes mod $n>0$. If h is a mapping of $V(G)$ into Q_n, its coboundary δh is the mapping of $E(G)$ into Q_n satisfying $\delta h(A)=\sum_a\eta(A, a)h(a)$ for each edge A, summed over all the vertices a, where $\eta(A, a)=0$ if A is a loop or if a is not an end of A, and $\eta(A, a)=1$ or -1 according as a is the positive or the negative end of A. If g is a mapping of $E(G)$ into Q_n, its boundary, denoted by ∂g, is the mapping of $V(G)$ into Q_n satisfying $\partial g(a)=\sum_A\eta(A, a)g(A)$. The number of mappings of $E(G)$ into Q_n which satisfy $\partial g(a)=0$ for all a and $g(A)\neq 0$ for all A is denoted by $\phi(G, n)$.

The four-colour conjecture is equivalent to each of the two propositions: (i) If G is a planar graph without a loop then $\theta(G, 4)>0$. (ii) If G is a planar graph without an isthmus then $\phi(G, 4)>0$. (An isthmus is an edge which does not belong to any circuit.) The author states two conjectures: (i) If G is any finite graph without an isthmus there exists an integer m such that $\phi(G, n)>0$ if $n\geqq m$. (ii) If G has no isthmus, $\phi(G, n)>0$ for $n\geqq 5$. He proves that if $\phi(G, n)>0$ then $\phi(G, n+1)>0$. A polynomial in two variables, which is more general than, but related to, $\theta(G, n)$ and $\phi(G, n)$, is also defined, and some of its properties are established. *G. A. Dirac* (London).

Burgess, C. E. Some theorems on n-homogeneous continua. Proc. Amer. Math. Soc. 5, 136–143 (1954).

Let M be a (compact metric) continuum. Suppose that n is the largest integer for which M is the union of n continua no one of which lies in the union of the others. Then M is indecomposable under index n. M is n-homogeneous if for each pair X, Y of sets, each of which consists of n points of M, there is a homeomorphism T, of M onto M, such that $T(X)=Y$. M is nearly n-homogeneous if for any n points $x_1, \cdots, x_n$ of M and any n open subsets $D_1, \cdots, D_n$ of M there exist points $y_1, \cdots, y_n$ of $D_1, \cdots, D_n$ respectively, and a homeomorphism T of M onto M such that

$$T(x_1\cup\cdots\cup x_n)=y_1\cup\cdots\cup y_n.$$

(It is not required that $T(x_i)=y_i$. Note that 1-homogeneity and 1-indecomposability are homogeneity and indecomposability in the usual sense.) If $H\subset M$, and each point x of H can be thrown into any open subset of M by a homeomorphism of M onto M, then M is nearly homogeneous over H. If $H\subset M$, and no proper subcontinuum of M contains H, then M is irreducible about H. The following are typical of the author's results: (I) Let H be a set consisting of n points of M $(n>1)$, such that M is irreducible about H, and such that M is nearly homogeneous over H. Then M is indecomposable under an index $r\leqq n$. (II) If M is nondegenerate, nearly 1-homogeneous, and irreducible about a

finite set, then M is indecomposable. (III) Let n be >1. If M is nearly 1-homogeneous and does not contain n points such that M is aposyndetic at each of them with respect to each of the others, then M is indecomposable. (IV) Let M lie in the plane E. Suppose that M is nearly 1-homogeneous, and that $E-M$ has only a finite number k of components C_i. Then M is the boundary of each C_i. And if $k>2$, then M is indecomposable. (V) If $M\subset E$ is nearly n-homogeneous ($n>1$), then M is either locally connected or indecomposable. (VI) If $M\subset E$ is n-homogeneous ($n>1$), then M is a simple closed curve. *E. E. Moise.*

Ramm, N. S., and Švarc, A. S. Geometry of proximity, uniform geometry and topology. Mat. Sbornik N.S. 33(75), 157–180 (1953). (Russian)

The authors give alternate proofs of theorems on proximity spaces due mainly to Smirnov [Doklady Akad. Nauk SSSR (N.S.) 84, 895–898 (1952); Mat. Sbornik N.S. 31(73), 543–574 (1952); these Rev. 14, 1107] as well as of some known results concerning compactifications of topological spaces. There are also some new results concerning mutual relations of uniform and "infinitesimal" (i.e., derived from proximity relation) notions such as continuous mappings, connected, discrete, bounded spaces. E.g., the following theorem is proved. If R, R_1 are proximity spaces with uniformities $\mathfrak{V}$, $\mathfrak{V}_1$, then an "infinitesimally" continuous mapping of R into R_1 is uniformly continuous provided that $\mathfrak{V}$ is maximal or $\mathfrak{V}_1$ is minimal. Reviewer's remark: it is to be noted that the theorem asserting the uniqueness of the metrizable uniformity compatible with a given metrizable proximity and proved by Efremovič [Mat. Sbornik N.S. 31(73), 189–200 (1952); these Rev. 14, 1106] appears also, essentially, in an article by Vilhelm and Vitner [Časopis Pěst. Mat. 77, 147–173 (1952); these Rev. 15, 641]. *M. Katětov* (Prague).

Efremovič, V. A., and Švarc, A. S. A new definition of uniform spaces. Metrization of proximity spaces. Doklady Akad. Nauk SSSR (N.S.) 89, 393–396 (1953). (Russian)

It is shown that any equivalence relation defined for directed sets of points of a set R and satisfying certain simple axioms may be obtained from exactly one uniformity $\mathfrak{B}$ on R in the following way: $\{x_\alpha\}\sim\{y_\alpha\}$ if and only if, for any $V\in\mathfrak{B}$, we have $(x_\alpha, y_\alpha)\in V$ for "large" α. In other words, uniform spaces may be defined in terms of equivalence of directed sets of points.

Let R be a proximity space (proximity relation denoted δ). Put $\{x_\alpha\}\sim\{y_\alpha\}$, where $\alpha\in A$, if, for any cofinal set $B\subset A$, $\{x_\alpha\}_{\alpha\in B}\delta\{y_\alpha\}_{\alpha\in B}$, where $\{x_\alpha\}_{\alpha\in B}$ denotes the set of all x_α, $\alpha\in B$. The relation $\sim$ satisfies the axioms mentioned above. The following metrization theorem is proved. A proximity space R is metrizable if and only if (1) $P\delta Q$ implies the existence of $x_n\in P$, $y_n\in Q$ with $\{x_n\}\sim\{y_n\}$, (2) if V denotes the collection of all $V\subset R\times R$ such that $\{x_n\}\sim\{y_n\}$ implies $(x_n, y_n)\in V$ for almost all n, then V contains a cofinal (with respect to the inclusion order) countable subcollection. *M. Katětov* (Prague).

Krishnan, V. S. On uniconvergence spaces. J. Madras Univ. Sect. B. 23, 174–181 (1953).

This note gives axioms for a notion of "uniconvergence in X over a directed system D". It is shown how a uniconvergence leads to a uniform structure in X, and conversely. *M. M. Day* (Urbana, Ill.).

Fan, Ky, and Struble, Raimond A. Continuity in terms of connectedness. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 161–164 (1954).

A single-valued mapping f of a topological space S into a topological space T is said to be connectedness-preserving if and only if it satisfies the following conditions: (i) for every connected subset A of S, the set $f(A)$ is connected; (ii) for every closed connected subset B of T, and for every point $x\in S-f^{-1}(B)$, there exists a neighborhood U of x which intersects only a finite number of components of $f^{-1}(B)$. It is clear that every continuous transformation is connectedness-preserving. The authors consider spaces satisfying the following conditions. Condition 1: The space has a basis, the complement of each element of which has a finite number of components. Condition 2: The space has a basis, the exterior of each element of which has a finite number of components. Condition 3: The space has a basis, the boundary of each element of which has a finite number of components. Condition 4: The space is rim-compact, i.e., it is a Hausdorff space and has a basis, the boundary of every element of which is compact. The paper proves that a connectedness-preserving transformation is continuous if any one of the following conditions hold: (1) T is a T_1-space satisfying Condition 1; (2) T is a regular space satisfying condition 2; (3) S is locally connected, and T is a T_1-space satisfying condition 3; (4) each of the spaces S and T is locally connected and T satisfies condition 4. *D. W. Hall* (College Park, Md.).

Katětov, Miroslav. On the dimension of non-separable spaces. I. Čehoslovack. Mat. Ž. 2(77) (1952), 333–368 (1953). (Russian. English summary)

The purpose of the paper is to prove, for arbitrary metric spaces, a series of results of dimension theory. Some of these results have already been reported without full proof [Doklady Akad. Nauk SSSR (N.S.) 79, 189–191 (1951); these Rev. 15, 145]. The first section of the paper is a systematic study of locally finite coverings; many of the results are known. In the second section, mappings into Banach spaces are considered. A map f of a metric space P into a metric space R is called uniformly 0-dimensional if given $\epsilon>0$ there exists $\eta>0$ such that if $S\subset P$ and diam $f(S)<\eta$ then S may be covered by open sets $\{H_\nu\}$ of diameter $<\epsilon$ and with no two of the H_ν intersecting. If P is a finite-dimensional metric space and R a Banach space with $\dim P\leqq\dim R$, then almost all (in the sense of category) bounded functions f from P to R are uniformly 0-dimensional. If $\dim P=\infty$, a similar result holds if R is suitably restricted. The main results are in the third section. It is proved that for a metrizable space P, the following are equivalent: (1) $\dim P\leqq n$, where $\dim P$ is the dimension in terms of open coverings; (2) $\operatorname{Ind} P\leqq n$, where $\operatorname{Ind} P$ is the inductive dimension defined in terms of separation of closed sets; (3) for any metric on P, there is a uniformly 0-dimensional map into E^n; (4) P may be represented as the union of $n+1$ 0-dimensional sets. Any metrizable space is homeomorphic with a dense subset of a complete metric space of the same dimension; also $\dim P\times Q\leqq\dim P+\dim Q$. *E. E. Floyd* (Charlottesville, Va.).

Mrówka, S. Solution d'un problème d'Urysohn concernant les espaces métriques universels. Bull. Acad. Polon. Sci. Cl. III. 1, 233–234 (1953).

(1) A metric space E is universal with respect to countable metric spaces if every countable metric space has an isometric image in E. (2) A metric space E is homogeneous

with respect to countable subsets if for every pair A, B of isometric countable sets there is an isometry of E onto E, carrying A onto B. It is shown that a space having both these properties cannot be separable. It follows that Urysohn's universal separable metric space U [Bull. Sci. Math. (2) 51, 43–64, 74–90 (1927)] does not have property (2). *E. E. Moise* (Ann Arbor, Mich.).

Hoheisel, Guido. Über Distanzfunktionen. Arch. Math. 5, 203–206 (1954).

Let M denote a non-empty abstract set, N a well ordered set, and $\rho(x, y)$ (denoted also by xy) a mapping of $M\times M$ into N. A function F on $N\times N$ to N is assumed, and it is supposed that $\rho(x, y)$ has the properties (i) for $x, y, z \in M$, $xy\leqq F(xz, yz)$ and (ii) for every $x \in M$, $\inf \{xz|z \in M\}=0$, where 0 is an element of N determined by F. The paper is concerned with the following question: What properties must F have in order that xy shall be a distance function (i.e., $xy\geqq 0$, $xy=0$ implies $x=y$, $xx=0$, and $xy=yx$)? It is found that if (α) $F(\xi, 0)\leqq\xi$, $F(0, \eta)\leqq\eta$ and (β) $\inf \xi=0$ implies $\inf F(\xi, \xi)=0$ (continuity of F at (0, 0)) then every $\rho(x, y)$ that satisfies (i), (ii) is a distance-function. A class of distance functions is determined by F, and $F(\xi, 0)=F(0, \xi)=\xi$ is a necessary condition that this class be non-null. *L. M. Blumenthal* (Leiden).

Keesee, John W. Sets which separate spheres. Proc. Amer. Math. Soc. 5, 193–200 (1954).

The result that if an n-manifold X is contained in an $(n+1)$-sphere Y, then $Y-X$ is the union of two disjoint, open, connected sets, each having X as point-set boundary, is here generalized by showing that the conclusion still holds if X is any closed subset of the $(n+1)$-sphere such that $H^n(X)$ is isomorphic to G and $H^n(A)=0$ for each closed proper subset A of X. In this notation, $H^n(X)$ is the n-dimensional cohomology group of X in some fixed cohomology theory defined on the category of compact pairs and satisfying the continuity property and the axioms of Eilenberg and Steenrod. G is the coefficient group, which can be an arbitrary Abelian group. *E. G. Begle.*

Hamstrom, Mary-Elizabeth. Concerning certain types of webs. Proc. Amer. Math. Soc. 4, 974–978 (1953).

Let M be a W_2-set in the plane [for definition, see Hamstrom, Trans. Amer. Math. Soc. 74, 500–513 (1953); these Rev. 14, 1109]. Let B be the boundary of M, and let J be the boundary of a component of the complement of M. An example is given, showing that J may contain as many as six limit-points of $B-J$. And it is proved that J cannot contain seven limit-points of $B-J$. *E. E. Moise.*

Bing, R. H. Locally tame sets are tame. Ann. of Math. (2) 59, 145–158 (1954).

A set A, lying in a triangulated space K, is locally tame if for each point x of A there is a (closed) neighborhood N of x in K, and a homeomorphism f of N into K, such that $f(N\cap A)$ and $f(N)$ are finite polyhedra. (If K is a manifold, this reduces to the usual definition, in which $f(N)$ is not required to be a polyhedron.) The author shows that (1) in a triangulated 3-manifold with boundary, every locally tame set is tamely imbedded, and (2) every 3-manifold K with boundary is triangulable; in fact, every triangulation of the boundary of K can be extended to give a triangulation of K. The proofs proceed mainly by explicit construction of homeomorphisms subject to boundary conditions of various kinds. *E. E. Moise* (Ann Arbor, Mich.).

Harrold, O. G., Jr., Griffith, H. C., and Posey, E. E. A characterization of tame curves in three-space. Proc. Nat. Acad. Sci. U. S. A. 40, 235–237 (1954).

A set M in Euclidean 3-space R^3 is locally polyhedral at a point x of R^3 if x has a closed neighborhood which meets M in a (possibly empty) finite polyhedron. An arc (or simple closed curve) J has property P if every point x of J has arbitrarily small neighborhoods bounded by topological 2-spheres K such that (1) K is locally polyhedral at each point of $K\cap(R^3-J)$, and (2) the number of points in $K\cap J$ is the order of x in J. J has property Q if for each point x of J there is a topological disk D such that (1) D is locally polyhedral at each point of $D\cap(R^3-J)$, (2) $G\cap J$ lies in the boundary of G, and (3) $G\cap J$ is a (closed) neighborhood of x in J. It is shown that if J has properties P and Q, then J is locally tamely imbedded, and conversely. *E. E. Moise* (Ann Arbor, Mich.).

Bagemihl, F. An extension of Sperner's Lemma, with applications to closed-set coverings and fixed points. Fund. Math. 40, 3–12 (1953).

Let $S_1, \cdots, S_m$ be a collection of n-simplexes such that $S_i\subset S_1$, $i=2, \cdots, m$; no S_i, $i=2, \cdots, m$, meets any face of S_1; $S_i\cap S_j=\phi$, $i, j>1$, $i\neq j$. The set $S_1[S_2, \cdots, S_m]$ obtained by deleting from S_1 the interior of each S_i, $i=2, \cdots, m$, is called an n-dimensional m-plex. Let the vertices of S_i be $(v_0^i, v_1^i, \cdots, v_n^i)$. If $S_1[S_2, \cdots, S_m]$ is simplicially subdivided, a function φ which assigns to each vertex w of the subdivision a number $\varphi(w)$ such that when w is on a simplex $(v^j_{i_0}v^j_{i_1}\cdots v^j_{i_k})$, then $\varphi(w)$ is one of the numbers $i_0, i_1, \cdots, i_k$, is called a vertex function for this subdivision. A simplex $(w_0w_1\cdots w_n)$ is called a representative simplex of the subdivision if $\varphi(w_i)=i$, and the number of representative simplexes is denoted by ρ. If each S_i is oriented, then π denotes the number of simplexes S_i, $i=2, \cdots, m$, which have the same orientation as S_1, and $\nu=m-1-\pi$.

The following results are derived. (a) If $\pi\neq\nu+1$, then $\rho>0$. If m is odd, then (even in the unoriented case) ρ is odd (and hence >0). The case $m=1$ is Sperner's Lemma. (b) Let $C_0, \cdots, C_n$ be closed sets such that every face $(v^j_{i_0}v^j_{i_1}\cdots v^j_{i_k})$ of the oriented $S_1, S_2, \cdots, S_m$ is contained in $C_{i_0}\cap\cdots\cap C_{i_k}$. Then if $\pi\neq\nu+1$, or if m is odd, $C_0\cap\cdots\cap C_n\neq\phi$. (c) Let f map $S_1[S_2, \cdots, S_m]$ into the Euclidean space containing it so that the frontier of each S_j is mapped into S_j. Then f has a fixed point if m is odd. (d) A similar generalization of the Kakutani fixed-point theorem. *E. G. Begle* (New Haven, Conn.).

Nikaidô, Hukukane. On von Neumann's minimax theorem. Pacific J. Math. 4, 65–72 (1954).

The von Neumann minimax theorem is proved in the following form: Let $K(x, y)$ be defined on the product space $X\times Y$, where X and Y are compact convex subsets of linear topological spaces, let $K(x, y)$ be continuous in each variable separately, let $K(x, y)$ be quasi-concave in x [$K(x_1, y)\geqq\lambda$, $K(x_2, y)\geqq\lambda$ implies $K(\alpha_1x_1+\alpha_2x_2, y)\geqq\lambda$, all λ, all $\alpha_i\geqq 0$, $\alpha_1+\alpha_2=1$] and quasi-convex in y. Then

$$\max_{x\in X}\min_{y\in Y} K(x, y)=\min_{y\in Y}\max_{x\in X} K(x, y).$$

It is interesting that the proof uses only the Brouwer fixed-point theorem and not the Kakutani generalization. *E. G. Begle* (New Haven, Conn.).

Wilder, R. L., and Roth, J. P. On certain inequalities relating the Betti numbers of a manifold and its subsets. Proc. Nat. Acad. Sci. U. S. A. 40, 207–209 (1954).

By analyzing certain exact homology sequences, using a field for coefficient group, and by using the Poincaré duality theorem for generalized manifolds, a number of relations between the Betti numbers of a closed subset and those of its complement are obtained. For example, if M is a closed subset of an orientable n-dimensional generalized manifold, S, then

(a) $$p^r(M) \leqq p^r(S) + p^{n-r-1}(S-M);$$

(b) $$p^r(S) \leqq p^r(M) + p^{n-r}(S-M);$$

(c) $$p^{n-r-1}(S-M) \leqq p^r(M) + p^{n-r-1}(S).$$

Also, if M is the boundary of an open subset U of S, then $p^r(M) \leqq p^r(\bar{U}) + p^{n-r-1}(U)$, and if U is ulc^{n-1}, then $p^r(M) \leqq p^r(\bar{U}) + p^{n-r-1}(\bar{U})$. In the above, $p^k(O)$, where O is open, is the Betti number based on compact cycles.

E. G. Begle (New Haven, Conn.).

Moore, John C. On homotopy groups of spaces with a single non-vanishing homology group. Ann. of Math. (2) 59, 549–557 (1954).

A space X is said to be of homology type (G, n) if it is arcwise-connected, simply-connected and if its singular homology groups are given by $H_r(G, n) = 0$, $r \neq 0, n$, $H_n(G, n) \cong G$. Then the homotopy groups of X do not depend on the choice of X within its homology type and we may write $\pi_r(X) = \pi_r(G, n)$. The author proves a Freudenthal suspension theorem for the groups $\pi_r(G, n)$, and obtains other results on these and associated groups which enable him further to elucidate the p-primary components of the homotopy groups of spheres. Thus, in particular, it is proved that, if p is odd, and n is odd, $n \geqq 5$, then the p-primary component of $\pi_q(S^n)$, written $\pi_q(S^n; p)$, is given by

$$\begin{aligned} \pi_q(S^n; p) &\cong Z_p, \quad q = n+2p-3,\ n+4p-5, \\ &= 0, \quad q \neq n+2p-3,\ n+4p-5,\ q < n+6p-8; \end{aligned}$$

also

$$\begin{aligned} \pi_q(S^3; p) &\cong Z_p, \quad q = 2p,\ 4p-3,\ 4p-2,\ 6p-5,\ 6p-4, \\ &= 0, \quad \text{otherwise},\ q < 8p-8. \end{aligned}$$

These special results extend previous results of the author [Ann. of Math. (2) 58, 325–350 (1953); these Rev. 15, 549]; they also overlap with results of Cartan and Serre [e.g., C. R. Acad. Sci. Paris 234, 393–395 (1952); these Rev. 13, 675], and unpublished (but 'well-known') results of Cartan. However, the general theorems in this paper are of such a kind that, roughly speaking, they can be applied whenever new information, from a different source, becomes available on homotopy groups of spheres.

There are very many misprints in this paper. Most serious, perhaps, is the error in the proof of Proposition 2.3 where (p. 554, l. 9.) "$\binom{j+2}{j}\alpha_{j+k}$" should read "$\binom{j+k}{j}\alpha_{j+k}$" and (l. 10) "$\binom{jp}{p} = j! \bmod p^k$ for any integer k" should, I believe, read "$\binom{jp}{p} \equiv j \bmod p$ for any integer j". Note also: on p. 553, the sentence before Proposition 2.3 should end "S_1^{2p+1}" not "S^{2p+1}", and, on p. 551, the proof of Theorem 1.3 should not contain the assertion that the mapping of D into Y^* induces isomorphisms of $H_q(D)$ with $H_q(Y^*)$ for $q \leqq m+2n+1$, since this is clearly false, in general, if $q = m+n+3$.

P. J. Hilton (Cambridge, England).

Wada, Hidekazu. Irrtümer. Tôhoku Math. J. (2) 5, 313 (1954).

See same J. (2) 4, 231–241 (1952); these Rev. 14, 894.

GEOMETRY

Bilo, J. On Schäfli sets of lines issued from the vertices of a simplex in a linear space of N dimensions. Simon Stevin 30, 1–4 (1954).

The author proves the two following propositions. I. Given a Schäfli set of lines l_i $(i = 0, 1, \cdots, n)$ passing through the corresponding vertices A^i of a simplex (A), when we project the lines of the set which pass through the vertices of any face $\alpha_{n-2}^{(pq)}$ of $n-2$ dimensions of (A), from this face on the opposite edge A^pA^q, the resulting $\frac{1}{2}n(n+1)(n-1)$ points $C_{(i)}^{(pq)}$ $(i = 0, 1, \cdots, p-1, p+1, \cdots, q-1, q+1;\ p, q = 0, 1, \cdots, n,\ p < q)$ or the harmonic conjugates $D_i^{(pq)}$ of these points with respect to the corresponding vertices A^p, A^q are the intersections of the edges of (A) with a primal of order $n-1$, according to whether n is odd or even. II. Given a Schäfli set of lines l_i $(i = 0, 1, \cdots, n)$ passing through the corresponding vertices A^i of a simplex (A) the $\frac{1}{2}n(n+1)(n-1)$ primes obtained by joining each face $\alpha_{n-2}^{(pq)}$ of $n-2$ dimensions of (A) to the lines of the set passing through the vertices of this face, are tangent primes of a primal of class $n-1$.

N. A. Court (Norman, Okla.).

Dias Agudo, Fernando Roldao. A new method for the study of plane sections of surfaces of the 2nd order. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) 2, 289–296 (1952). (Portuguese. French summary)

Bagchi, Hari Das, and Sarkar, Shib Sankar. The relationship between harmonic and anharmonic collineations in a plane. Amer. Math. Monthly 61, 397–401 (1954).

***Delone, B. N. Kratkoe izloženie dokazatel'stva neprotivorečivosti planimetrii Lobačevskogo.** [**A short exposition of the noncontradictoriness of the planimetry of Lobačevskiĭ.**] Izdat. Akad. Nauk SSSR, Moscow, 1953. 128 pp. 4.65 rubles.

According to the preface the first two chapters of the book serve the purpose of supplying amateurs with a schooling of something between grammar and high school, who "not infrequently" still invent proofs of the parallel axiom, with a simple explanation of why such a proof is impossible.

The introduction stresses the importance of Lobačevskiĭ. It was new to the reviewer that he, like Gauss, had investigated the possibility of the astronomical space not being euclidean, and that he, in objecting to Kant, made remarks of the following type: There is no reason to take it for granted that molecular mechanics is governed by euclidean geometry. There are some purely ornamental quotations from Lenin and Stalin; the views expressed by them coincide with those of any intelligent person.

Chapter I gives a complete system of axioms for absolute geometry following, but improving considerably in exactness, the approach of F. Schur's based on motion as basic concept. Only few of the implications of the axioms are carried out. Chapter II supplies the Klein model of the hyperbolic plane. Since the author does not wish to use projective geometry, he obtains the hyperbolic motion as the spatial affinities which carry a circular cone into itself. It is shown that the hyperbolic plane possesses the degree of mobility required by Schur's axioms. This approach is

very amusing even to the expert. These affinities are further exploited in Chapter III and IV to actually develop hyperbolic geometry. Chapter V and two appendices discuss Poincaré's model, the pseudo-sphere and the relations to special relativity. *H. Busemann* (Copenhagen).

Neculcea, M. Extension de l'axiome de congruence des triangles dans le plan. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 693–699 (1952). (Romanian. Russian and French summaries)

One of the axioms of congruence in Hilbert's Grundlagen der Geometrie [7. Aufl., Teubner, Leipzig, 1930] postulates that if any two triangles are such that two sides and the included angle of one triangle are congruent to two sides and the included angle of the other, then the two triangles have each pair of corresponding angles congruent. The author shows that this assumption may be weakened by excepting from its provisions a non-empty set T of triangles. The set T may even be infinite provided each finite region of the plane contains only finitely many of its members.
L. M. Blumenthal (Leiden).

Fadini, Angelo. Su particolari piani affini generalizzati. Ricerca, Napoli **5**, no. 1–2, 57–64 (1954).

The author considers a special case of the generalized affine spaces of R. Permutti [Ricerche Mat. **2**, 192–203 (1954); these Rev. **15**, 735], namely that of a projective plane with two lines removed. The equations of various transformations are determined. Distance between points is defined by the logarithm of their cross-ratio with the points where the line joining them cuts the excluded lines. Its properties are examined; in particular, there is a line along which distances are zero, and the algebraic sum of the sides of any triangle is zero. The angle between lines is also defined, but in a less usual manner since the natural generalization is found meaningless. If the excluded lines are real, then through any proper point there are two parallels to a given line not passing through the intersection of the excluded lines, and parallelism is not transitive. Perpendiculars are in general unique. If the excluded lines are imaginary, then distance is determined to within a multiple of a constant; parallels in general do not exist. *P. M. Whitman.*

Zappa, Guido. Sui piani grafici finiti h-l-transitivi. Boll. Un. Mat. Ital. (3) **9**, 16–24 (1954).

Following Baer [Amer. J. Math. **64**, 137–152 (1942); these Rev. **3**, 179] a plane is called h-l transitive if, given fixed lines l and h and any points U, P_1, P_2 on h where U may be the intersection of l and h, but not P_1 or P_2, there exists a collineation fixing U and the points of l and also mapping P_1 onto P_2. Here U, P_1, and P_2 uniquely determine the collineation. Taking l as the line at infinity the h-l collineations form a group doubly transitive on the finite points of h in which only the identity fixes two points. It is shown that a subgroup H_t fixing a finite point of h is cyclic if and only if the plane is Desarguesian.
Marshall Hall, Jr. (Columbus, Ohio).

Lombardo-Radice, Lucio. Sui sistemi cartesiani di coordinate dei piani grafici h-l-transitivi. Boll. Un. Mat. Ital. (3) **9**, 24–29 (1954).

Following Baer as in the paper reviewed above, it is shown that the coordinates in an h-l transitive plane form a Dickson ring in which addition is an abelian group, and multiplication forms a group. Lines are given by equations $y=xm+b$, and the left distributive law $r(s+t)=rs+rt$. Here the collineations can be determined by the mappings $(x, y)\rightleftarrows(rx, ry+s)$. *Marshall Hall, Jr.*

Lombardo-Radice, Lucio. Piani grafici finiti a coordinate di Veblen-Wedderburn. Ricerche Mat. **2** (1953), 266–273 (1954).

It is shown that the construction of finite Veblen-Wedderburn systems is equivalent to finding a family of p^t-1 automorphisms σ_x of an elementary abelian group of order p^t forming a loop according to a product $\sigma_x\circ\sigma_y=\sigma_u$, and satisfying a one-sided distributive ring with respect to this product, addition being from the elementary abelian group.
Marshall Hall, Jr. (Columbus, Ohio).

Zappa, Guido. Sui piani grafici finiti transitivi e quasitransitivi. Ricerche Mat. **2** (1953), 274–287 (1954).

A finite projective plane is transitive if it has a group G of collineations transitive on its points. G is called regular if the only element fixing a point is the identity. Such planes include the cyclic planes studied by the reviewer [Duke Math. J. **14**, 1079–1090 (1947); these Rev. **9**, 370]. With $n+1$ points on a line, if $n\equiv1 \pmod 3$, then $N=n^2+n+1=3u$ and there is a group of order N which is non-abelian but has a normal subgroup which is cyclic of order u. If n is a prime power, the corresponding Desarguesian plane has such a collineation group. The group with $n=10$ does not yield a plane. A quasi-transitive group G is a collineation group transitive on the finite points, as in a Veblen-Wedderburn plane, or on all but one finite point as in the cyclic affine planes of A. J. Hoffman [Canadian J. Math. **4**, 295–301 (1952); these Rev. **14**, 196]. *Marshall Hall, Jr.*

Edge, W. L. Geometry in three dimensions over GF(3). Proc. Roy. Soc. London. Ser. A. **222**, 262–286 (1954).

This is a detailed discussion of the three-dimensional geometry with 40 points. Particular study is made of the quadric surfaces, the hyperboloids, equivalent to $x^2+y^2+z^2+t^2=0$, and the ellipsoids, equivalent to $x^2+y^2+z^2-t^2=0$. This geometry is of particular interest with respect to the simple group of order 360 which is simultaneously the alternating group on six letters LF$(2, 3^2)$ and PO$_2(4, 3)$.
Marshall Hall, Jr. (Columbus, Ohio).

Karzel, Helmut. Erzeugbare Ordnungsfunktionen. Math. Ann. **127**, 228–242 (1954).

In the n-dimensional projective space S_n over a division ring R let the hyperplanes h correspond to left-homogeneous vectors $h=[su_0, \cdots, su_n]$ and the points P to right-homogeneous vectors $P=(x_0t, \cdots, x_nt)$. Here P lies on h if the scalar product hP is zero. An order function is any function $f(h, P)$ in S_n, defined for all hyperplanes h and points P which is zero if P lies on h and either $+1$ or -1 otherwise. A generated order function is one which depends solely on the scalar product bP, each hyperplane h and point P being normalized to be represented by a definite single vector. It is shown here that a generated order function can be characterized in a purely geometric manner. Thus, if P_1 and P_2 are two points given as normalized vectors, every hyperplane through the point P_1-P_2 assigns the same order value to both P_1 and P_2. This observation is the starting point for the proof. *Marshall Hall, Jr.*

Busemann, Herbert. Motions with maximal displacements. Comment. Math. Helv. **28**, 1–8 (1954).

Let R be a G-space [i.e., a metric space in which geodesics with most of the usual geometric properties exist; see the

author, Trans. Amer. Math. Soc. 56, 200–274 (1944); these Rev. 6, 97]. Designate by $\rho(p)$ a positive continuous function (which exists in every G-space) such that, for any x, y in $S(p, \rho(p))$ (the sphere of radius $\rho(p)$ and center p), there exists a point z such that (xyz); i.e., such that $xy+yz=xz$, where the distance between x and y is denoted by xy. Writing $z\Phi$ for the image of the point z under the mapping Φ, the author proves: if Φ is a motion of the G-space R which is not the identity, and if $zz\Phi=\sup_{x\in R} xx\Phi<\frac{1}{2}\rho(z)$, then $(zz\Phi z\Phi^2)$. As the introduction states, the implications of this "nearly trivial remark" are surprisingly strong. Among the most interesting we quote: (1) A closed group of motions of a compact G-space R is a Lie group. If the group Γ of all motions which R possesses is transitive, then R is a topological manifold and $\dim \Gamma \leqq \frac{1}{2}(\dim R)(\dim R+1)$. (2) A one-parameter group of motions of a compact G-space possesses an orbit which is a geodesic. (3) Let R' be a compact G-space with an abelian fundamental group and a straight (i.e., $\rho(p)\equiv\infty$) universal covering space. Then the closed geodesics in any free homotopy class of R' have the same length and cover R' simply. No geodesic in R' has multiple points. The paper concludes with a characterization of Minkowski geometry. The methods are those of metric geometry, except, of course, for (1), where appeal is made to the structure of compact groups. *L. W. Green.*

Kelly, P. J. Barbilian geometry and the Poincaré model. Amer. Math. Monthly 61, 311–319 (1954).

Barbilian space [Časopis Pěst. Mat. Fys. 64, 182–183 (1935)] is obtained by assigning to each two points a, b of the interior K of a simple closed, plane curve J the number

$$d(a, b)=\log\left[\max_{p\in J}(pa/pb)\right]+\log\left[\max_{q\in J}(qb/qa)\right]$$

as distance, where xy denotes the euclidean distance of points x, y. [A brief investigation of this space is given by the reviewer, Univ. Missouri Studies 13, no. 2 (1938).] The author establishes some known properties of Barbilian space, discusses Barbilian geodesics, and defines a generalization in which K and J are replaced, respectively, by any planar set and a closed set J disjoint from it.

L. M. Blumenthal (Leiden).

Convex Domains, Extremal Problems, Integral Geometry

Hammer, P. C. Diameters of convex bodies. Proc. Amer. Math. Soc. 5, 304–306 (1954).

Dans E_n une famille continue et extérieurement simple de droites (c'est-à-dire telle qu'il en passe une et une seule par toute point extérieur à une sphère de rayon assez grand) n'est pas nécessairement la famille des diamètres d'un corps convexe pour $n>2$. La preuve est donnée à partir de la remarque suivante: lorsqu'un point intérieur à un corps convexe est tel que toute corde qui y passe est un diamètre du corps, c'est un centre de symétrie. *J. Favard.*

Knothe, Herbert. Zur Theorie der konvexen Körper im Raum konstanter positiver Krümmung. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) 2, 336–348 (1952).

Let S be the surface area, and M the surface integral of the mean curvature, of the surface of a convex body K in Euclidean 3-space. Further let R' denote the maximum radius of circles inscribed to orthogonal projections of K, let R'' denote the corresponding minimum radius for circumscribed circles, and write $\lambda=R''-R'$. It is known that $\lambda>0$ and that $M^2-4\pi S\geqq(2\pi\lambda)^2$. The author seeks analogous results for a convex body K in non-Euclidean 3-space of constant curvature unity. With the help of Clifford's definition of parallels, he gives similar meanings to the symbols and shows that the inequalities $\lambda>0$, $M^2+S^2-4\pi S\geqq(2\pi\lambda^*)^2$ are valid, with $\lambda^*=\tan(\lambda/2)$, provided that at each boundary point of K there is a supporting sphere of radius $\pi/4$.

L. C. Young (Madison, Wis.).

Rešetnyak, Yu. G. An extremal problem from the theory of convex curves. Uspehi Matem. Nauk (N.S.) 8, no. 6(58), 125–126 (1953). (Russian)

In a plane with a Minkowski metric M defined by choosing for the unit circle a convex curve K with centre, let $\pi(M)$ denote, by analogy with the usual definition of π, half the length in this metric of the curve K. Then $3\leqq\pi(M)\leqq4$ and the extremes are attained, respectively, for a regular hexagon and for a square. *L. C. Young.*

Fejes Tóth, L. On close-packings of spheres in spaces of constant curvature. Publ. Math. Debrecen 3 (1953), 158–167 (1954).

The author makes the following two conjectures. (1) The greatest possible density for a packing of spherical 3-space by more than two equal spheres is

$$12\left(1-\frac{5}{\pi}\sin\frac{\pi}{5}\right)=0.774\cdots,$$

obtained by the 120 inscribed spheres of the cells of the spherical honeycomb $\{5, 3, 3\}$, in which each sphere touches twelve others [Coxeter, Scripta Math. 18, 113–121 (1952), p. 116; these Rev. 14, 494]. (2) The greatest possible density for a packing of hyperbolic 3-space by equal spheres (of finite or infinite radius) is

$$(1+1/2^2-1/4^2-1/5^2+1/7^2+1/8^2-\cdots)^{-1}=0.853\cdots,$$

attained by the inscribed horospheres of the cells of the hyperbolic honeycomb $\{6, 3, 3\}$, in which each horosphere touches infinitely many others [Coxeter and Whitrow, Proc. Roy. Soc. London. Ser. A. 201, 417–437 (1950), p. 426; these Rev. 12, 866]. He supports these conjectures by two theorems: one giving an upper bound for the number of spheres of given radius that can be packed round a sphere of the same size, and the other giving a lower bound for the volume of an n-hedron containing a given sphere.

H. S. M. Coxeter (Toronto, Ont.).

Molnár, J. Ausfüllung und Überdeckung eines konvexen sphärischen Gebietes durch Kreise. II. Publ. Math. Debrecen 3 (1953), 150–157 (1954).

The author gives a careful and ingenious proof that every covering of a convex region on a sphere by equal small circles has greater density than $2\pi/3\sqrt{3}=1.209\cdots$, which is the density of the thinnest covering of the Euclidean plane [L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, Springer, Berlin, 1953, p. 58; these Rev. 15, 248]. The analogous packing problem was solved in part I [Publ. Math. Debrecen 2, 266–275 (1952); these Rev. 14, 988].

H. S. M. Coxeter (Toronto, Ont.).

Kon-Fossen [Cohn-Vossen], S. Nonrigid closed surfaces. Uspehi Matem. Nauk (N.S.) 9, no. 1(59), 63–81 (1954). (Russian)

Translated from Math. Ann. 102, 10–29 (1929).

Algebraic Geometry

Stavropoulos, Pothitos. Sur les diamètres rectilignes des courbes algébriques planes d'ordre $2\nu+1$. Bull. Soc. Math. Grèce 28, 115–127 (1954). (Greek. French summary)

Il est bien connu que H. Lebesgue a demontré que "le plus grand nombre des diamètres d'une courbe algébrique irréductible du degré $\nu\geqq 3$ est ν quand ν est impair, et $\nu+2$ quand ν est pair. Il y a une exeption là seulement pour les nombres $\nu=6$, 8, 12, 16, 20, 24". Dans le présent travail nous avons examiné le même problème pour ν impair d'une méthode analytique et nous terminons aux résultats suivants: 1. La direction conjuguée d'une diamètre d'une courbe algébrique irréductible d'ordre impair est la direction asymtotique de celle-ci. 2. Une courbe algébrique irréductible d'ordre ν impair peut avoir le plus ν diamètres. De plus, avec les formules 24–25 du §3 nous déterminons les coefficients de chaque diamètre. *Résumé de l'auteur.*

Fava, Franco, e Parodi, Francesco Alberto. Coppie di elementi differenziali curvilinei riferiti isometricamente per proiezione. Univ. e Politecnico Torino. Rend. Sem. Mat. 12, 145–157 (1953).

Given two curves in a euclidean plane and a point on each, how is it possible by projection to get an isometry of a certain order in a neighborhood of the given points? Similar questions for space curves. *E. Bompiani* (Rome).

Semple, J. G. Some investigations in the geometry of curve and surface elements. Proc. London Math. Soc. (3) 4, 24–49 (1954).

G. Gherardelli has shown [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 821–828 (1941); these Rev. 8, 222] that the differential elements of the second order E_2 of a projective plane can be represented unexceptionally on the points of a non-singular fourfold M_4^{330} of a space S_{40}. A new derivation of this representation is given. A similar method is applied to find unexceptional models of the elements E_2 of S_r (in particular, through a point). A hint of similar researches on surface elements (calottes or caps) is given. No mention is made of the extensive literature on the subject. *E. Bompiani* (Rome).

Gallarati, Dionisio. Sul numero dei complessi algebrici di rette, di ordine assegnato, che contengono una data rigata algebrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 14, 213–220 (1953).

Let F be an algebraic ruled surface in S_r and K an algebraic line complex in S_r of order m. A. The maximum number of linearly independent K's containing the lines of F is at most

$$\theta_m=\frac{1}{m+1}\binom{m+r}{r}\binom{m+r-1}{r-1}-m(r-1)-1.$$

B. For $m>1$ the maximum is reached if and only if F is a rational normal surface or a cone whose section is a rational normal curve in S_{r-1}. C. For $m=1$ the maximum is reached if and only if F is a rational normal surface or a cone of any sort. *R. J. Walker* (Ithaca, N.Y.).

Gallarati, Dionisio. Sulle varietà di S_r composte di ∞^1 S_k, i cui S_k appartengono al massimo numero di complessi lineari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 14, 408–412 (1953).

The results stated in the previous review are generalized, for the case $m=1$, by allowing K to be a complex of S_k's in S_r, and F a V_{k+1}, the locus of ∞^1 S_k's. The θ_m of Theorem A is replaced by

$$\theta(r,k)=\binom{r+1}{k+1}-r+k-1,$$

and Theorem C generalizes directly: for a maximum, F is either rational normal or consists of S_k's through a common S_{k-1}. As a special case we have D: If the S_k's of F osculate a curve C, then $\theta(r,k)$ can be replaced by

$$\lambda(r,k)=\binom{r+1}{k+1}-(k+1)(r-k)-1,$$

and the maximum is reached only if C is rational normal. *R. J. Walker* (Ithaca, N. Y.).

Gallarati, Dionisio. Sul massimo numero di complessi lineari di S_k di S_r linearmente indipendenti, ai quali appartengono gli S_k tangenti di una V_k di S_r. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 15, 10–15 (1953).

The maximum $\lambda(r,k)$ of Theorem D of the preceding review also applies to the case where F consists of the ∞^k tangent S_k's of a variety V_k in S_r. The maximum is reached only if V_k is a rational normal V_k^{r-k+1}. *R. J. Walker.*

d'Orgeval, Bernard. A propos des surfaces de genres 1 et de rang 2. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 40, 109–117 (1954).

A propos des surfaces algébriques de genres 1 et de rang 2 (c'est-à-dire surfaces dont tous les genres sont égaux à l'unité et qui représentent une involution du second ordre appartenant à une surface également de genres 1) on connait un théorème de L. Godeaux, qui donne des conditions nécessaires et suffisantes pour qu'une surface F de genres 1, d'ordre $2\pi-2$, normale dans un espace S^π, soit de rang 2: il faut que F possède huit points doubles coniques et que parmi les hyperquadriques de S^π passant par les huit points doubles il en existe un système linéaire $\infty^{\pi-3}$ touchant F tout le long d'une courbe Λ d'ordre $2\pi-2$ et genre $\pi-2$ (ce système étant le seul dans ces conditions). L'auteur démontre ici que la surface F n'existe pas toutes les fois que l'on peut écrire π sous la forme $3+4\sum n_i^2$ $(i=1,\cdots,\alpha)$, la somme contenant au plus quatre nombres n_i differents de zéro. Il traite avant tout le cas simple où $\alpha=1$, $n_1=1$; puis le cas $\alpha=1$, n_1 quelconque; et enfin le cas général. Dans tous les cas, en supposant l'existence de la surface F, on trouverait sur F deux systèmes différents de courbes Λ; d'où l'impossibilité de l'existence de F. Dans la démonstration l'auteur se sert des résultats qu'il a obtenu à ce sujet dans sa thèse [Paris, 1945]. *E. Togliatti* (Gênes).

Molinari, Anna Maria. Il gruppo delle trasformazioni cremoniane di S_n immagini delle proiettività dell'S_1 n-complesso. Ricerca, Napoli 5, no. 1–2, 65–72 (1954).

Roth, Leonard. On elliptic threefolds. Rend. Circ. Mat. Palermo (2) 2 (1953), 141–158 (1954).

An elliptic threefold V is characterised by the fact that it contains a congruence Γ of birationally equivalent elliptic curves and an elliptic pencil of surfaces (not generated by curves of Γ) which are non-singular and birationally equivalent. Threefolds of this type may be regarded as generalisations of elliptic surfaces, and, like these, admit a continuous ∞^1 group of self-transformations. If the canonical system on such a threefold is pure, it is compounded of the congruence Γ, and (when effective) consists of elliptic surfaces; more-

over, the linear genus and arithmetic genus of V are each equal to unity. The author proceeds to develop a number of properties of elliptic threefolds, and to show how they may be constructed. The classification of such threefolds depends on the determinants of the threefold, that is, the number of intersections of a surface of the pencil with a curve of the congruence Γ. *J. A. Todd.*

Roth, Leonard. On threefolds which contain congruences of elliptic curves. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **12** (1953), 387–425 (1954).

This paper continues the study already begun by the author [Ann. Mat. Pura Appl. (4) **34**, 247–276 (1953); these Rev. **14**, 1117] of threefolds of linear genus unity. Two classes of threefolds V are considered in the present paper. The first contains a congruence Γ of elliptic curves, with the property that any surface belonging to Γ is elliptic. Such a threefold contains a pencil $|C|$ (rational or elliptic) of surfaces not in Γ, free from base points. If $|C|$ is elliptic, then V is an elliptic threefold. If $|C|$ is rational, then any reducible surfaces belonging to $|C|$ must be multiple surfaces, and V either has $P_g=P_i=0$ or else contains a virtual canonical surface of order zero. In this case the curves of Γ are all irreducible. Among threefolds of this class are the Abelian threefolds, already studied by the author in the paper reviewed above and the improperly Abelian threefolds (which map the superficially irregular involutions carried by Picard varieties). The second class of threefolds likewise contains a congruence Γ and a pencil $|C|$, and admits a representation in a multiple threefold with a branch surface consisting of images of surfaces of $|C|$ together with surfaces in Γ. All the threefolds considered have their canonical and pluricanonical surfaces contained in Γ, and accordingly have linear genus unity. An important part of the paper is devoted to explicit construction of threefolds of the various types considered. *J. A. Todd.*

Benedicty, Mario. Sopra le trasformazioni birazionali in sè di un campo neutro a sostegno ellittico o iperellittico. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11** (1952), 411–433 (1953).

L'Autore considera un campo neutro γ di caratteri p, δ_1, δ_2 (cioè una curva algebrica C di genere effettivo p con fissate δ_1 coppie neutre di punti distinti e δ_2 di punti coincidenti) il cui genere virtuale è dunque $\pi=p+\delta_1+\delta_2$ e determina quei campi neutri a curva sostegno ellittica od iperellittica che ammettono trasformazioni birazionali in sè (ossia trasformazioni birazionali di C che mutano coppie neutre in coppie neutre); il caso del genere effettivo $p=0$ è stato studiato dall'Autore in una precedente ricerca [Ann. Scuola Norm. Super. Pisa (3) **4**, 157–173 (1950); questi Rev. **13**, 272]. Tutti i campi γ_G che ammettono un gruppo finito di trasformazioni birazionali in sè si ottengono fissando un gruppo finito G di trasformazioni birazionali in sè di C e considerando come coppie neutre di γ_G un sistema di coppie che sia trasformato in sè da G. Sono possibili le seguenti coppie (e le loro trasformate per operazioni di G): coppia generica di punti distinti, coppia generica di una involuzione di G, coppia di punti uniti (distinti) di una medesima operazione di G, coppia generica di punti coincidenti, coppia di punti coincidenti in un punto unito per una delle operazioni di G. Nel caso $p=1$ l'Autore considera tutti i possibili gruppi finiti di trasformazioni birazionali su una curva ellittica a modulo generale o particolare ed ottiene numerosi tipi di campi γ_G dall'esame dei quali risulta, indicando con ν e λ rispettivamente l'ordine di G ed il periodo delle sue trasformazioni, $\nu\leqq 4(\pi-1)$, $\lambda\leqq 2(\pi-1)$ se C è a modulo generale; $\nu\leqq 8(\pi-1)$, $\lambda\leqq\max[4, 2(\pi-1)]$ se C è armonica;

$$\nu\leqq 6(\pi-1),\quad \lambda\leqq\max[6, 2(\pi-1)]$$

se C è equianarmonica, ed esempi provano come tali limitazioni siano effettivamente raggiunte.

Se C è una curva iperellittica di genere $p\geqq 2$ rappresentata su una retta doppia X con $2p+2$ punti di diramazione Δ_j mediante la g_2^1 esistente su C, il gruppo G_ν subordina su X un gruppo finito H_μ di proiettività che mutano in sè l'insieme dei punti Δ_j e dei punti A_i', B_i', A_l' corrispondenti in X ai punti delle coppie neutre (A_iB_i), (A_lA_l) e viceversa fissato su X un insieme di punti Δ_j (in numero pari $\geqq 6$) che sia mutato in sè da un gruppo finito di proiettività resta indotto un gruppo finito $G'_{\nu'}$ di trasformazioni birazionali sulla curva iperellittica C individuata dai punti di diramazione Δ_j. Fissate su C $\delta=\delta_1+\delta_2$ coppie neutre, il cui sistema sia mutato in sè da $G'_{\nu'}$ o da un suo sottogruppo G_ν si ottiene un campo γ di genere virtuale $\pi=p+\delta$ con ν trasformazioni birazionali in sè. Risulta (se $\delta\geqq 1$) $\nu\leqq f(\pi)$ e $\lambda\leqq 4\pi-2$ (essendo $f(\pi)=8(\pi-1)$ per ogni $\pi\geqq 3$ diverso da 5, 6, 11, 12, 13, 14, 15, mentre $f(5)=f(6)=48$; $f(11)=\cdots=f(15)=120$) ed esistono infiniti valori di π per i quali le limitazioni sono raggiunte. Tenendo presenti i risultati ottenuti dall'Autore per $p=0, 1$ si ha che un campo neutro γ in senso stretto ($\delta\geqq 1$) sopra una curva razionale, ellittica, iperellittica avente genere virtuale $\pi>1$ e diverso dal campo χ di caratteri $p=\delta_1=0$, $\pi=\delta_2=2$ possiede $\nu\leqq f_2(\pi)$ trasformazioni birazionali in sè (ove $f_1(\pi)=8(\pi-1)$ per ogni $\pi\geqq 2$ diverso da 3, 5, 6, 11, 12, 13, 14, 15, mentre $f_1(3)=24$, $f_1(5)=48$, $f_1(6)=60$, $f_1(\pi)=120$ per $\pi=11, \cdots, 15$) ed esistono infiniti valori di π per i quali la limitazione è raggiunta. Per campi neutri in senso lato ($\delta\geqq 0$) distinti da χ, si ha $\nu\leqq f_2(\pi)$ (ove $f_2(\pi)=48$ per $\pi=2, 3, 4$; $f_2(\pi)=120$ per $\pi=5, \cdots, 13$; $f_2(\pi)=8(\pi+1)$ per $\pi\geqq 14$). Infine il periodo di una trasformazione birazionale in sè di un campo neutro di genere virtuale $\pi>1$ su una curva razionale, ellittica, iperellittica, e diverso da χ, è $\lambda\leqq 4\pi-2$.

D. Gallarati (Genova).

Benedicty, Mario. Sui caratteri di matrici quasi abeliane equivalenti. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **12** (1953), 332–339 (1954).

Sia ω una matrice quasi abeliana (m.q.a.) di caratteri p, δ_1, δ_2, ρ, ossia una matrice equivalente ad un'altra avente la forma normale di Severi [Funzioni quasi abeliane, Pont. Acad. Sci. Scripta Varia, v. 4, 1947; questi Rev. **9**, 578], intendendo che due matrici ω, ω' siano equivalenti se $\omega'=\alpha\omega C$ con α complessa non degenere e C intera unimodulare. È chiaro che una matrice equivalente ad una m.q.a. è una m.q.a., ed è noto che una matrice isomorfa ad una m.q.a. è essa stessa quasi abeliana e riducibile ad una forma normale avente gli stessi caratteri, intendendo che ω ed ω' siano isomorfe se $\omega'=\alpha\omega C$ con α complessa e C razionale, entrambe non degeneri. Qualora le forme normali di una data m.q.a. avessero tutte i medesimi caratteri, tali caratteri potrebbero essere attribuiti alla ω, ma ciò non accade e quindi si pone il problema di determinare le forme normali cui è equivalente (o isomorfa) una data m.q.a. o almeno di determinare i caratteri di tali forme normali. L'Autore risolve completamente quest'ultimo problema in alcuni casi particolari ed assegna delle condizioni sufficienti affinchè una m.q.a. sia equivalente ad una per cui il carattere p sia minore; a tale scopo stabilisce dapprima che se ω è una matrice normale di Severi, esiste una matrice normale di Severi ω' ad essa equivalente (isomorfa) avente gli stessi

caratteri e per la quale la relativa matrice normale di Riemann è una prefissata matrice normale di Riemann ad essa equivalente (isomorfa). *D. Gallarati* (Genova).

Rosati, Mario. Osservazioni su alcuni gruppi finiti di omografie appartenenti ad una varietà di Picard e ad una varietà abeliana di rango due. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11** (1952), 453–469 (1953).

L'Autore dimostra dapprima che ogni sistema lineare completo $|A|$ sopra una varietà di Picard W_p, rappresentato dall'equazione $\vartheta_{u_p}(u+c)=0$ è mutato in sè da un gruppo finito G di $2\delta^2$ trasformazioni birazionali appartenenti alla W_p, essendo δ il determinante delle funzioni intermediarie $\vartheta_{u_p}(u)$; o, in forma proiettiva, ogni modello proiettivo normale della W_p di Picard di un $S_{\delta-1}$, immagine del sistema lineare $\infty^{\delta-1}$ (supposto semplice) $\vartheta_{u_p}(u+c)=0$ è una varietà d'ordine $\delta\cdot p!$ invariante per un gruppo finito di omografie, generato da $2p+1$ di esse. Per $p=1$ si riottiene il ben noto teorema secondo cui una curva ellittica normale C^l di S_{l-1} è invariante per un gruppo finito di $2l^2$ omografie, dotato di tre elementi generatori. L'Autore considera poi una varietà abeliana V_p di rango 2 immagine dell'involuzione generata su W_p da una trasformazione di prima specie, per esempio $u'\equiv -u$ (mod ω): essa possiede un gruppo abeliano finito di 2^{2p} trasformazioni birazionali di equazioni $u'\equiv u+\sigma$ (mod ω) essendo σ un arbitrario semiperiodo; con la determinazione di quelle trasformazioni che mutano in sè un dato sistema lineare di ipersuperficie di V_p si ottengono i gruppi di omografie appartenenti ad un dato modello proiettivo di V_p.

D. Gallarati (Genova).

Rosati, Mario. Sopra le funzioni abeliane pari e le varietà abeliane di rango due. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, 28–61 (1952).

Sia K un corpo di funzioni abeliane di p variabili $u_1, u_2, \cdots, u_p$, ω una matrice di Riemann relativa a K, $\delta_1, \delta_2, \cdots, \delta_p$ i divisori elementari di una matrice principale M associata ad ω, W_p la varietà di Picard del corpo K. W_p possiede un gruppo continuo assolutamente transitivo di trasformazioni birazionali in sè, $u'\equiv -u+c$ (mod ω) (trasf. di 1ª sp.), $u'\equiv u+c$ (mod ω) (trasf. di 2ª sp.), e la classificazione delle involuzioni di second'ordine appartenenti a W_p e birazionalmente distinte equivale a quella dei tipi birazionalmente distinti di trasformazioni birazionali involutorie di W_p in sè. Le trasformazioni di 2ª specie sono involutorie solo se c è un semiperiodo, e quindi (escludendo il semiperiodo nullo) W_p possiede $2^{2p}-1$ involuzioni I_2'' generate da trasf. di 2ª specie. La $\bar{V}_p$ immagine di una siffatta I_2'' è birazionalmente equivalente alla varietà di Picard relativa ad un altro corpo K' di funzioni abeliane; le trasf. di 1ª specie sono tutte involutorie e le varietà immagini di involuzioni I_2' da esse generate sono tutte birazionalmente equivalenti e posseggono un gruppo abeliano d'ordine finito 2^{2p} di trasformazioni birazionali involutorie. Il loro studio riducesi a quello della V_p di rango 2 immagine dell'involuzione $u'\equiv -u$ (mod ω) ed il corpo delle funzioni razionali su V_p si identifica col sottocorpo di K delle funzioni abeliane pari. Il problema di rappresentare sotto forma trascendente le ipersuperficie della V_p e di parametrizzare V_p in uno spazio proiettivo è ricondotto a quello, risolto dall'Autore, della determinazione di tutte le funzioni intermediarie associate ad un dato corpo K di funzioni abeliane che danno come quoziente le funzioni abeliane pari di K, annullando le quali si può rappresentare una qualunque ipersuperficie non eccezionale esistente su V_p. L'Autore determina i sistemi lineari completi di ipersuperficie algebriche esistenti su V_p, e la dimensione di tali sistemi relativamente ad una distribuzione qualunque dei divisori elementari; tale dimensione dipende dal numero γ $(0\leqq\gamma<p)$ dei numeri pari esistenti tra gli interi δ_p/δ_j $(j=1, 2, \cdots, p)$.

D. Gallarati (Genova).

Conforto, Fabio. Sulle funzioni abeliane singolari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 754–759 (1953).

L'Autore dà una semplice dimostrazione del teorema fondamentale per le funzioni abeliane singolari [G. Scorza, Mem. Mat. Sci. Fis. Nat. Soc. Ital. Sci. (3) **19**, 139–183 (1916)], che assegna una condizione necessaria e sufficiente affinchè il corpo di funzioni abeliane definito da una matrice di Riemann $\omega(p, 2p)$ di genere p (>1) abbia l'indice di singolarità k $(\geqq 1)$, ossia esistano k e non più sistemi nulli razionali della ω linearmente indipendenti (intendendo per sistema nullo reale (o razionale) di ω ogni matrice reale (o rispettivamente razionale) ed emisimmetrica M, per cui $\omega M\omega_{-1}=0$). Allo scopo l'Autore mostra che se $\Omega=\left\|\begin{matrix}\omega\\ \bar{\omega}\end{matrix}\right\|$ è matrice non degenere può porsi una corrispondenza di isomorfismo tra lo spazio vettoriale S dei sistemi nulli reali di ω e lo spazio S'_{p^2} delle matrici hermitiane $L(p, p)$. La totalità I' di tutte le L definite e positive—a cui corrisponde l'insieme I dei sistemi nulli reali e principali di ω (per i quali cioè $i\omega M\bar{\omega}_{-1}$ è definita positiva)—è un insieme aperto e convesso; lo stesso avviene dunque per I. E si ha che ω è matrice di Riemann di genere p se e solo se Ω è non degenere ed in S esiste un punto razionale interno ad I; ed affinchè ω abbia indice di singolarità k occorre e basta che in I penetri un $S_k\subset S$ passante per l'origine e razionale.

D. Gallarati (Genova).

Rev

***Dantoni, Giovanni. Metodi geometrici per lo scioglimento delle singolarità delle superficie e varietà algebriche.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 99–112. Casa Editrice Perrella, Roma, 1953.

A presentation, with minor extensions and elaborations, of earlier results of the author [Ann. Scuola Norm. Super. Pisa (3) **5**, 355–365 (1951); these Rev. **13**, 865]. Some of the methods used by others in attacking the problem of reducing the singularities of surfaces are discussed.

R. J. Walker (Ithaca, N. Y.).

Segre, Beniamino. Nuovi metodi e resultati nella geometria sulle varietà algebriche. Ann. Mat. Pura Appl. (4) **35**, 1–127 (1953).

The object of this paper is to develop a general theory of algebraic varieties, using as basic tool the theory of systems of equivalence of sub-varieties, in a way similar to that in which linear systems of curves were used by Castelnuovo, Enriques and Severi in their birational theory of surfaces.

The paper begins with a brief account of the theory of systems of equivalence, as developed by Severi, the author, and Todd. Then the most important new idea is introduced. On a non-singular variety V, of dimension v, let P be an irreducible non-singular sub-variety of dimension p, and let $A_1, \cdots, A_s$ be sub-varieties of V, of dimension $a_1, \cdots, a_s$, all passing through P. If $\sum a_i-(s-1)v=q\leqq p$, the varieties $A_1, \cdots, A_s$ will, in general, meet in P and in a residual variety Q, of dimension q, and it is shown that if $A_1, \cdots, A_s$ vary in systems of equivalence and continue to pass through P, then Q belongs to a well-defined system of equivalence,

denoted by $(A_1\cdots A_s)_V{}^P$. It is then shown that if $A_1, \cdots, A_s$ are each of dimension $v-1$, and if $V_i{}^s(A)$ denotes the sum of the intersections of i of the varieties A_{ji} selected in all possible ways from $A_1, \cdots, A_s$ without repetition, it is possible to define systems of equivalence $P_{V,i}$ $(i=0, \cdots, p)$ uniquely on V by the equations

$$(A_1\cdots A_s)_V-(A_1\cdots A_s)_V{}^P=\sum_{i=0}^{t} P_{V,i}\cdot V^s{}_{t-i}(A) \qquad (s=V-P, \cdots, V),$$

where $t=s-p+v$, and $P\cdot Q$ denotes the ~~vertical~~ virtual intersection of P, Q, and it is shown that the $P_{V,i}$ depend only on P and V, and not on $A_1, \cdots, A_s$, so that they are covariants of immersion of P in V. An argument, which the reviewer is unable to follow completely, is given to show that the $P_{V,i}$ determine well-defined systems of equivalence on P.

The rest of this very long paper is devoted to the study of the geometry of a variety P embedded in V, using the covariants of immersion $P_{V,i}$. Even a mere list of the problems considered would be extremely lengthy, and it may be more convenient if only a few typical examples are given. (1) If $P\subset M\subset V$, to determine the covariants of immersion of P in V in terms of those of P in M and of M in V. (2) If P, M are sub-varieties of V with regular intersection P', to determine the covariants of immersion of P' in M in terms of those of P in V. (3) To determine the covariants of immersion of the intersection of subvarieties of V in terms of the covariants of these subvarieties. (4) To determine the intersection of $(A_1\cdots A_s)_V{}^P$ with P in terms of the covariants of P, $A_1, \cdots, A_s$. (5) To extend the theory to the case in which $A_1, \cdots, A_s$ pass multiply through P. (6) To determine the locus of singularities imposed on a variety Q by making it pass through P. (7) Given s copies $V_1, \cdots, V_s$ of the fundamental variety V, and varieties $M_1, \cdots, M_s$ in $V_1, \cdots, V_s$ respectively, to determine the covariants of immersion of $M_1\times M_2\times\cdots\times M_s$ in $V_1\times V_2\times\cdots\times V_s$ in terms of those of M_i in V_i.

From this last problem, the paper goes on to consider the covariants of immersion of the diagonal variety of $V_1\times\cdots\times V_s$ in the direct product. In this way it is possible to get a new definition of the canonical systems on V, and to obtain the theorem of adjunction. (This definition of the canonical systems appears to have close connections with known, but unpublished, methods of investigating the Stiefel-Whitney classes on a manifold.)

The processes and operations used in this paper will prove of very great importance in the theory of varieties, but unfortunately the methods employed seem to the reviewer to leave many of the proofs incomplete, and there are a number of mistakes in the paper (though, as far as the reviewer can see, none of these is vital). Here are some examples. On p. 9, the definition $M\subset V'$ is carefully restricted to the case in which V' is effective, but on p. 20, it is used, without explanation, in the case in which V' is virtual. It is asserted that if P varies in a system of equivalence on V, and $A_1, \cdots, A_s$ vary in systems of equivalence subject to the condition that each A_i contains P, then $(A_1\cdots A_s)_V{}^P$ varies in a system of equivalence; according to the definition of equivalence given, a rational cubic C and an elliptic cubic Γ in S_3 are equivalent, yet if A_1, A_2, A_3 are cubic surfaces in S_3, $(A_1A_2A_3)_{S_3}^{C}$ consists of 10 points and $(A_1A_2A_3)_{S_3}^{\Gamma}$ consists of 12 points. The results of §16′ are false, but the reviewer understands that the author has an alternative treatment of these topics, which in any case do not form an integral part of the argument.

W. V. D. Hodge (Cambridge, England).

Chow, Wei-Liang. The Jacobian variety of an algebraic curve. Amer. J. Math. 76, 453–476 (1954).

Let C be an algebraic curve (of genus g and order h) defined over a field k, and let K be any extension of k. Denoting by $G_a(C)$ the Abelian group of all divisors of degree zero in C, and by $G_l(C)$ the subgroup of the principal divisors (which correspond to the virtual sets of points of C linearly equivalent to zero), the author gives a precise definition of the Jacobian varieties, by saying that an algebraic variety J defined over K is a Jacobian variety of C if it has the following four properties: (1) J is an Abelian variety defined over K; (2) there is a homomorphic mapping Φ (called the canonical homomorphism) of $G_a(C)$ onto J, whose kernel is $G_l(C)$; (3) Φ is rational, and can be defined "over K" in a certain convenient sense; (4) Φ has the "universal mapping" property, i.e., if Ψ is any rational homomorphism of $G_a(C)$ into any Abelian variety A, then Ψ is the product of Φ and a rational homomorphism of J into A.

The existence of J is well known when k is the complex field (and a transcendental proof in this case is here sketched in no. 2). In the general case, the problem of existence was proposed and partially solved by A. Weil [Variétés abéliennes et courbes algébriques, Hermann, Paris, 1948, §V; these Rev. 10, 621], with the construction of the Jacobian variety as an "abstract" variety. There remained however to see whether J exists as a projective algebraic variety in the usual sense, and whether both J and Φ can be defined over k. These important questions are answered here in the affirmative, by proceeding as follows.

First of all, by the use of the associated forms, the divisors of C of a sufficiently high degree n (say $n>2g-2$) are represented by points of a variety C^n, of dimension n. Every divisor class of C of degree n corresponds to a subvariety G of dimension $n-g$ in C^n: and the ∞^g varieties G thus obtained are then represented by points of a new variety V, defined over k, by using the associated forms again.

The varieties G constitute an involutional system on C^n: it is proved (no. 5) that they have all the same order $(=h^{n-g})$ and that C^n is non-singular, whence follows the fundamental fact that also V is non-singular, by applying the main theorem of a previous paper [Chow, Amer. J. Math. 72, 247–283 (1950), p. 258; these Rev. 11, 615]. The use of this theorem could be avoided, by considering a derived normal model W of V, since it is shown directly (no. 4) that W is non-singular and related to V by a one-to-one correspondence without exceptions: this alternate proof has the advantage of simplicity, but there is the possibility that W is not defined over k, except when k is perfect. Finally it is shown (no. 6) that the transformation of C^n onto V generates in a natural way a rational homomorphism Φ having the required properties, so that V (or, alternatively, W) is a Jacobian variety of C. *B. Segre.*

Rosenlicht, Maxwell. Generalized Jacobian varieties. Ann. of Math. (2) 59, 505–530 (1954).

Soient K un corps de fonctions algébriques d'une variable sur un corps de base k, et $\mathfrak{o}$ un sous anneau semi local de K. L'auteur étudie la "jacobienne généralisée" $J(\mathfrak{o})$, c'est à dire une variété algébrique isomorphe au quotient du groupe des diviseurs de degré 0 sur un modèle C de K qui ne contiennent aucune des places dont l'anneau contient $\mathfrak{o}$, par le sous groupe des diviseurs (f) des fonctions f de K qui sont inversibles dans $\mathfrak{o}$ [cf. Rosenlicht, Ann. of Math. (2) 56, 169–191 (1952); ces Rev. 14, 80]. Dans le cas classique où

k est le corps des nombres complèxes, l'existence de $J(\mathfrak{o})$ résulte de généralisations du théorème d'Abel et du théorème d'inversion de Jacobi: c'est le quotient du groupe additif $(k)^{\pi}$ ($\pi=\mathfrak{o}$-genre de K; cf. loc. cit.) par un sous groupe discret de périodes (lequel n'est pas en général de rang maximum, ce qui fait que $J(\mathfrak{o})$ n'est pas en général compacte). Dans le cas abstrait l'auteur a besoin de divers résultats sur les variétés de groupes; l'un complète un théorème d'A. Weil relatif à l'existence d'une variété de groupe G birationnellement équivalente à une variété V munie d'une loi de composition normale, et montre que la construction de G peut se faire sans agrandir le corps de base k, pourvu que V ait suffisament de points rationnels sur k; un autre décrit des données permettant de construire un homomorphisme d'une variété de groupe dans une autre; un troisième détermine le sous groupe algébrique engendré par une sous variété d'une variété de groupe. Ceci étant, on construit $J(\mathfrak{o})$ en définissant une loi de composition normale sur le produit symétrique $C^{(\pi)}$ et l'existence d'une application canonique $\varphi_{\mathfrak{o}}$ de C dans $J(\mathfrak{o})$ établit l'isomorphisme ci dessus décrit. Quand $\mathfrak{o}$ et $\mathfrak{o}'$ sont deux sous anneaux semi locaux de K tels que $\mathfrak{o}\subset\mathfrak{o}'$, il existe un homomorphisme canonique $\tau_{\mathfrak{o}',\mathfrak{o}}$ de $J(\mathfrak{o})$ sur $J(\mathfrak{o}')$ tel que $\tau_{\mathfrak{o}',\mathfrak{o}}\circ\varphi_{\mathfrak{o}}=\varphi_{\mathfrak{o}'}$; le noyau H de $\tau_{\mathfrak{o}'\mathfrak{o}}$ est un sous groupe algébrique et une sous variété rationnelle de $J(\mathfrak{o})$, et $J(\mathfrak{o})$ est birationnellement (mais non birégulièrement en général) équivalent à $J(\mathfrak{o}')\times H$. Avec certaines hypothèses sur $\mathfrak{o}$ et $\mathfrak{o}'$, en particulier que les corps quotients de $\mathfrak{o}'$ sont isomorphes au corps de base, H est birégulièrement isomorphe à un groupe de la forme $(G_m)^t\times H_1$, où G_m est le groupe multiplicatif du domaine universel, et où H_1 est un groupe porté par un espace affine; en caractéristique nulle H_1 est même de la forme $(G_a)^s$, où G_a est le groupe additif du domaine universel. Quand $\mathfrak{o}'$ est la clôture intégrale de $\mathfrak{o}$, $J(\mathfrak{o}')$ est la jacobienne classique de C, et est le plus grand quotient de $J(\mathfrak{o})$ qui soit une variété abélienne. Ces résultats paraissent être de précieux guides pour l'étude générale de la structure des variétés des groupes. *P. Samuel.*

Igusa, Jun-ichi. On the arithmetic normality of the Grassmann variety. Proc. Nat. Acad. Sci. U. S. A. **40**, 309–313 (1954).

This paper gives a new proof valid for arbitrary characteristics, that the Grassmann variety of r-spaces in projective n-space is arithmetically normal. The theorem is then used to prove Severi's theorem that any positive divisor on a Grassmann variety is the complete intersection with a hypersurface in the ambient space, and that the Chow form of any irreductible variety of dimension r in S_n (an arbitrary field) is a homogeneous polynomial in Grassmann coordinates of the generic $(n-r-1)$-space of S_n.
W. V. D. Hodge (Cambridge, England).

Differential Geometry

Jonas, Hans. Bemerkungen zu dem nach Tissot benannten Abbildungsproblem. Math. Nachr. **11**, 187–189 (1954).

The problem studied is that of mapping the plane onto itself while preserving areas and carrying an orthogonal network of curves (u, v) through the point (x, y) into the lines parallel to the rectangular coordinate axes. By using a contact transformation this problem has been reduced to the solution of the partial differential equation $\theta_{\alpha\beta}=\theta$. [See A. Voss, Enzykl. Math. Wiss., Bd III 3, Teubner, Leipzig, 1903, pp. 355–440, esp. p. 372; A. Korkine, Math. Ann. **35**, 588–604 (1890).] In this paper a new simple reduction to $\theta_{\alpha\beta}=\theta$ is presented, the principal result being that given an integral θ of $\theta_{\alpha\beta}=\theta$, x, y and u, v can all be stated in terms of θ in a form free of quadratures. *A. Schwartz.*

Finzi, Arrigo. Errata: Sur le problème de la génération d'une transformation donnée d'une courbe fermée par une transformation infinitésimale. Ann. Sci. Ecole Norm. Sup. (3) **70**, 403–404 (1953).

See same Ann. (3) **69**, 371–430 (1952); these Rev. **14**, 685.

Verenskiold, W. Theory of map projections. Bull. Géodésique **1953**, 153–161 (1953). (German, Spanish, French, Italian summaries)

An elementary discussion is presented developing the differential equations for conformal and equal area map projections of an ellipsoid. The paper represents an abridgement of the author's monograph: Kartprojeksjoner, Oslo, 1945. *N. A. Hall* (Minneapolis, Minn.).

Tanturri, Giuseppe. Caratterizzazione geometrica di alcuni sistemi ∞^5 di curve sopra una V_3. Univ. e Politecnico Torino. Rend. Sem. Mat. **12**, 177–194 (1953).

Regarding u, v, w as coordinates in a three-dimensional manifold immersed in an n-dimensional projective space, the author studies non-metrical characterizations of the five-parameter families of curves defined by systems of equations of the form

$$v'''=c(u, v, w, v', w', v''), \quad w''=\beta(u, v, w, v', w', v''),$$

where the primes denote differentiation with respect to u. [For other closely related work by the author, see same Rend. **9**, 145–172 (1950); **10**, 243–258 (1951); these Rev. **12**, 610; **13**, 682.] Consider the ∞^1 curves which have a common given tangent at a given point. It is shown that if, for every choice of the given point and given tangent, the second order elements of these curves at the given point belong to a quadric hypersurface in the projective space, the function β in the defining system of equations is linear in the variable v''. Several other more complicated results of this kind are obtained, giving geometrical interpretations of further specializations of the functions c and β. In conclusion, a few particular cases, serving to illustrate the general results, are discussed in detail. *L. A. MacColl.*

Krishna, Shri. On the reciprocal congruence of Ribaucour. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 1094–1111 (1953).

Une congruence rectiligne de Ribaucour peut être définie à partir de deux surfaces S et S_1, se coupant avec orthogonalité des éléments linéaires, en menant par chaque point de S_1 la parallèle à la normale au point correspondant de S; S_1 est la surface de référence de la congruence, S la surface directrice.

Si l'on intervertit les roles de S et S_1 dans la construction précédente, on obtient une nouvelle congruence, qui est précisément la "congruence réciproque de Ribaucour" étudiée par l'auteur (et définie par lui et Mishra). L'étude est basée sur la considération des fonctions caractéristiques ϕ et ϕ_1 de Bianchi relatives à l'une ou l'autre des deux congruences réciproques. L'auteur commence par former l'expression du tenseur symétrique fondamental de S_1; il en déduit un certain nombre de relations entre ϕ, ϕ_1, l'angle θ des normales à S, S_1, les éléments d'aire correspondants

de S, S_1, leurs représentations sphériques et la surface S_0 associée à S, et le rayon vecteur r de S_0. Il s'étend ensuite plus spécialement sur le cas où la congruence réciproque est normale, ce qui lui permet d'énoncer diverses propriétés géométriques intéressantes. *P. Vincensini* (Marseille).

Jha, P., and Chariar, V. R. A note on rectilinear congruences. Math. Student 21 (1953), 81–86 (1954).

L'auteur établit, par une nouvelle méthode, trois théorèmes antérieurs de R. S. Mishra [Rev. Fac. Sci. Univ. Istanbul (A) 16, 95–101 (1951); ces Rev. 13, 278] relatifs aux couples de surfaces réglées d'une congruence rectiligne dont les lignes de striction sont sur la surface de référence et indique quelques nouvelles propriétés des couples de surfaces envisagées par Mishra lorsque la congruence rectiligne envisagée est normale. Ces propriétés concernent l'applicabilité des couples en question (qui est une condition nécessaire et suffisante pour que la congruence donnée soit normale), où leurs points de contact (qui sont les foyers de la congruence normale). L'auteur montre aussi que si la surface de référence de la congruence normale considérée est coupée obliquement par la congruence, et si les lignes de striction de deux surfaces réglées de la congruence sont sur la surface de référence, les deux surfaces étant applicables, les directions orthogonales aux rayons sur la surface de référence bissectent l'angle des lignes de striction.

P. Vincensini (Marseille).

Jonas, Hans. *W*-Strahlensysteme mit einem pseudosphärischen Brennflächenmantel, Guichardsche Kongruenzen und isometrische Vosssche Flächen. Math. Nachr. 11, 105–128 (1954).

The author considers W-congruences with a pseudospherical focal surface. It is shown that such a W-congruence determines uniquely a congruence of Guichard. A plane normal to the line of this congruence of Guichard and which divides in constant ratio the segment between the focal points, envelopes a surface of Voss (F). To each F a surface F' exists obtainable without quadratures and which is a transform of F by a deformation preserving the geodesic conjugate system. Finally, transformations of a surface of Voss are considered which include the system of surfaces obtained from F by deformations mentioned above.

J. Haantjes (Leiden).

Fernandez, German. Ruled developable surfaces in 4-dimensional spaces of constant curvature. Univ. Nac. Eva Peron. Publ. Fac. Ci. Fisicomat. Serie Segunda. Revista 4, 508–525 (1953). (Spanish)

A la suite d'un travail de Santaló [Publ. Inst. Mat. Univ. Nac. Litoral 4, 3–42 (1942); ces Rev. 4, 170] l'auteur forme un système de conditions nécessaires et suffisantes pour qu'une 2-surface réglée, plongée dans un espace à courbure constante de dimension 4 soit développable au voisinage d'une génératrice; voir aussi Yung-Chow Wong [Trans. Amer. Math. Soc. 59, 467–507 (1946); ces Rev. 7, 529]. L'auteur utilise les formalismes et méthodes d'Elie Cartan et en donne, dans une première partie, un bref et élégant exposé. *A. Lichnerowicz* (Paris).

Bompiani, Enrico. Un teorema del Bianchi sulle rigate applicabili. Boll. Un. Mat. Ital. (3) 9, 1–4 (1954).

La publication récente du 2ème volume des oeuvres de L. Bianchi [Perrella, Roma, 1953, pp. 48–95; ces Rev. 15, 591] a fourni à l'auteur l'occasion de présenter sous un aspect nouveau l'une des toutes premières recherches de l'illustre géomètre italien. C et C' étant deux courbes gauches, Bianchi a montré que les deux surfaces réglées engendrées par les perpendiculaires aux normales principales de C et C' issues des points courants correspondants (relatifs à la même valeur de l'arc u) P et P' de C et C' faisant l'angle constant θ avec les courbes, sont applicables si les rayons de courbure et de torsion de C et C' verifient la relation (1) $\cos\theta/R+\sin\theta/T=\cos\theta/R'+\sin\theta/T'$. L'auteur considère les deux éléments du 3ème ordre E_3 et E_3' des deux courbes C, C' ayant pour centres P et P', et considère (1) comme définissant un angle θ par l'égalité

$$\tan\theta=-(1/R-1/R')/(1/T-1/T'),$$

qu'il appelle "angle de Bianchi" relatif aux éléments E_3 et E_3' et dont il donne une construction à partir des deux éléments. Si l'on reprend la construction des surfaces réglées envisagées par Bianchi à partir de deux courbes gauches quelconques C et C', en substituant à l'angle constant qui y intervient l'angle de Bianchi $\theta(u)$ relatif aux différents couples d'éléments E_3 et E_3' correspondant à la même valeur u de l'arc, les deux surfaces réglées pbtenues sont applicables. On voit d'ailleurs aussi-tôt que C et C' sont les lignes de striction des deux surfaces réglées, et que la condition nécéssaire et suffisante pour l'applicabilité de telles surfaces est qu'elles aient la même courbure gaussienne aux points correspondants de leurs lignes de striction C et C'. L'auteur termine par l'indication d'un certain nombre de problèmes de géométrie euclidienne ou riemannienne directement inspirés par l'extension qu'il a donnée au théorème de Bianchi. *P. Vincensini* (Marseille).

Backes, F. Sur certains réseaux et leur rapport avec les surfaces minima. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 40, 118–124 (1954).

L'auteur expose une généralisation intéressante de la notion de réseaux conjugués sur une surface Σ. Il substitue au plan tangent au point courant P de Σ une sphère tangente S de rayon ρ, montre que, si (du, dv), $(\delta u, \delta v)$ définissent respectivement un déplacement infiniment petit quelconque sur Σ et le déplacement correspondant dans la direction de la tangente au cercle caractéristique de S, ces deux déplacements sont liés par l'équation

$$(E-\rho L)du\delta u+(F-\rho M)(du\delta v+dv\delta u)+(G-\rho N)dv\delta v=0$$

(où $E, \cdots, N$ sont les coefficients des deux premières formes fondamentales de Σ), et appelle réseau R_ρ tout réseau de Σ dont les directions des tangentes en chaque point sont liées par l'équation précédente. Tout réseau de Σ est un réseau R_ρ pour une valeur convenable de ρ, et, si $\rho=R_1+R_2$, R_1 et R_2 étant les rayons de courbure principaux de Σ en P, la sphère S donne naissance à une infinité de réseaux R_ρ de Σ, dits par l'auteur réseaux spéciaux. Ces réseaux spéciaux ont pour images sphériques des réseaux orthogonaux, et comme tout réseau orthogonal d'une sphère est l'image des lignes de courbure d'une surface (définie à une homothétie près), on voit apparaître une correspondance par plans tangents parallèles entre un réseau R_ρ spécial et le réseau de courbure d'une certaine surface (M). L'auteur étudie cette correspondance. Il montre en particulier que si (M) est minima on peut obtenir une infinité de sphères S engendrant des réseaux spéciaux ayant pour image sphérique le réseau (isotherme) image de ses lignes de courbure, les sphères engendrant deux quelconques de ces réseaux restant homothétiques dans une homothétie de rapport fixe et dont le centre est constamment le point M correspondant de la surface minima (M). Réciproquement d'ailleurs, si deux

réseaux spéciaux ont meme image sphérique et si les rayons des sphères qui les engendrent sont en rapport constant, un centre de similitude des deux sphères décrit une surface minima. *P. Vincensini* (Marseille).

Pinl, M. Über die Gausssche Krümmung der reellen Minimalflächen im R_4. Monatsh. Math. 58, 27–32 (1954).

It is proved that in Euclidean four-dimensional space there are no real minimal surfaces with constant non-vanishing Gaussian curvature. *S. B. Jackson.*

Moór, A., und Török, A. Über zwei Extremaleigenschaften des Kreisbogens und der Kugelfläche. Acta Sci. Math. Szeged 15, 157–163 (1954).

Let T_1OT_2 be a triangle with $OT_1=OT_2$. Among all the arcs of curve with continuous curvature $\kappa(s)$ which lie inside the triangle, the arc of circle gives the max min $\kappa(s)$ and the min max $\kappa(s)$. In 3-space an analogous result holds for the curvature of Gauss and the mean curvature of the convex surfaces tangent to a cone of revolution along the basis and contained in it. *L. A. Santaló* (Buenos Aires).

Wintner, Aurel. On the local role of the theory of the logarithmic potential in differential geometry. Amer. J. Math. 75, 679–690 (1953).

Une métrique définie positive $ds^2=g_{\alpha\beta}du^\alpha du^\beta$ sur un ouvert D (simplement connexe, suffisamment petit) d'un plan (u^1, u^2), est dite de classe C^n si ses coefficients sont de classe C^n. Pour $n\geqq 2$ cette métrique admet une courbure K de classe C^{n-2}. L'auteur se propose d'etudier la classe importante des C^2-métriques admettant une courbure de classe C^1. Il montre qu'étant donnée, dans l'espace ordinaire, une surface par un plongement de classe C^3, la métrique induite n'est de classe C^2 que si la courbure est de classe C^1. Au sujet des C^2-métriques sur D, il établit à l'aide d'un de ses propres résultats [même J. 73, 569–575 (1951); ces Rev. 13, 384] le théorème suivant: toute C^2-métrique admettant une courbure de classe C^1 est isométrique (localement) à une C^2-métrique conforme (au plan euclidien) et ayant ellememe une courbure de classe C^1. Toute C^1-application locale réalisant cette isométrie est nécessairement de classe C^3. Plus généralement, toute C^n-métrique $(n>1)$ admettant une courbure de classe C^{n-1+m} $(0\leqq m\leqq 2)$ est isométrique à une C^{n+m}-métrique conforme. Les démonstrations, outre les techniques de Lichtenstein et H. Weyl, utilisent les caractères de différentiabilité d'un potentiel logarithmique, d'où le titre du papier. Celui-ci se termine par des raffinements intéressants relatif à l'introduction de conditions de Hölder uniformes (localement). *A. Lichnerowicz* (Paris).

Terracini, Alessandro. Sulle coppie di rami con la stessa origine e gli stessi spazi osculatori. Univ. e Politecnico Torino. Rend. Sem. Mat. 12, 265–281 (1953).

Suppose that in a projective space S_{n+1} of $n+1$ $(n\geqq 2)$ dimensions two curves C, C' have the same osculating spaces $\omega_1, \cdots, \omega_\alpha$ of dimensions $1, \cdots, \alpha$, respectively, for any fixed $\alpha(1\leqq\alpha\leqq n)$ at a common point O, and that nonhomogeneous projective coordinates in S_{n+1} are chosen so that the expansions representing C, C' in the neighborhood of O are

$$C:\quad x_i=\sum_{k=0}^{\infty}a_{i,\,i+k}x^{p_i+k}\qquad (a_{ii}\neq 0,\ i=1, \cdots, n),$$

$$C':\quad x_i=\sum_{k=0}^{\infty}a'_{i,\,i+k}x^{p_i+k}\qquad (a'_{ii}\neq 0,\ i=1, \cdots, n),$$

where p_i are positive integers satisfying $1<p_1<p_2<\cdots<p_n$. If there exists an integer $m\geqq 0$ such that $a_{i,\,i+k}=a'_{i,\,i+k}$ $(1\leqq i\leqq n,\ 0\leqq k\leqq m)$ and that $a_{i,\,i+m+1}-a'_{i,\,i+m+1}$ are not all zero, then $\gamma_{ij}=z_i/z_j$ $(1\leqq i\leqq n,\ 1\leqq j\leqq n,\ i\neq j)$ are projective invariants, where

$$(*)\qquad z_i=(a_{i,\,i+m+1}-a'_{i,\,i+m+1})/a_{ii}.$$

Geometric interpretations of these projective invariants γ_{ij} for a general n as well as $n=2$ are given. B. Segre [Quart. J. Math., Oxford Ser. 17, 35–38 (1946); these Rev. 7, 393] has studied the case where $p_i=i+1$ $(i=1, \cdots, n)$ and $a_{ii}-a'_{ii}$ are not all zero. *C. C. Hsiung.*

Terracini, Alessandro. Relazioni tra invarianti proiettivi duali di coppie di elementi curvilinei. Boll. Un. Mat. Ital. (3) 8, 368–374 (1953).

Let z_i^* $(1\leqq i\leqq n)$ correspond to z_i defined by equations (*) in the above review under the dualistic transformation which converts each point of a curve in S_{n+1} into the osculating hyperplane of the curve at the point. In this paper it is shown that $\rho z_i^*=\sum_{r=1}^{n}h_{ir}z_r$ $(1\leqq i\leqq n)$, where h_{ir} are constants depending only on n, m and p's. *C. C. Hsiung.*

Dalmasso, Liana. Particolari terne di curve sghembe in geometria proiettiva differenziale. Boll. Un. Mat. Ital. (3) 9, 66–73 (1954).

In a recent paper [Math. Z. 56, 409–442 (1952); these Rev. 14, 687], Barner has determined an invariant parameter for a certain pair of curves in an ordinary projective space S_3. Three curves in S_3 are called an associated triple of the first kind (and of the second kind otherwise), if their points are in a correspondence such that any three corresponding points are collinear and if the three pairs of curves determined by them have the same invariant parameter of Barner. Two curves in S_3 are called an associated pair of the first kind if they are in an associated triple of the first kind. In this paper, necessary and sufficient conditions are obtained for two twisted cubics as well as two director curves on a ruled surface in S_3 to be an associated pair of the first kind. Moreover, it is shown that in S_3 for a given director curve Γ of a ruled surface there exist ∞^2 other ruled surfaces whose director curves form associated pairs of the first kind with the curve Γ. *C. C. Hsiung* (Bethlehem, Pa.).

Picasso, Ettore. Alcune osservazioni sull'uso delle connessioni proiettive nello studio delle superficie di S_4. Rend. Sem. Fac. Sci. Univ. Cagliari 23, 1–8 (1953).

By means of projective connections Bortolotti [Rend. Circ. Mat. Palermo 56, 1–57 (1932)] has established the theory of a hypersurface S_{n-1} as well as a variety X_m of m dimensions immersed in a projective space S_n of n $(>m)$ dimensions. In this paper, by applying the general method of Bortolotti to a surface X_2 in a projective space S_4 the author deduces some known results for X_2, which were obtained by the method based on differential forms and equations. *C. C. Hsiung* (Bethlehem, Pa.).

Petrescu, Şt. De la classification des espaces à connexion projective P_2. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 4, 29–37 (1952). (Romanian. Russian and French summaries)

The projective connexion of a two-dimensional space with torsion but with projective curvature zero can be brought into a canonical form if it admits a Pfaffian of class one. Spaces of this kind can be classified by means of their group of automorphisms. The last section contains results for the case with torsion zero. *J. A. Schouten* (Epe).

Petrescu, Şt. Considérations sur les automorphismes des espaces A_2 à connexion affine symétrique. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 3, 149–155 (1951). (Romanian. Russian and French summaries)

In a former note [same Bul. 2, 639–646 (1950); these Rev. 14, 1123] it was proved that the L_2's can be classified according to the number of parameters (4, 3 or 2) of their group of automorphisms. Here the same problem is discussed for A_2's ($A_2 = L_2$ without torsion). The same three classes exist but the canonical expressions of the parameters $\Gamma^h{}_{ji}$ have other forms. The first two classes admit an invariant Pfaffian of class 1 and class 2 respectively. *J. A. Schouten.*

Vrănceanu, G. Propriétés globales des espaces à connexion affine. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 3 (1951), 451–455 (1952). (Romanian. Russian and French summaries)

It is asked whether an A_n with a symmetric and integrable connexion is globally equivalent to the ordinary E_n, especially in the cases when the $\Gamma_{ji}{}^h$ are special functions or constants. In the last case there is a commutative and associative algebra with n independent by hypercomplex numbers associated with the A_n. By means of this algebra a canonical form can be established. *J. A. Schouten.*

Tachibana, Syun-ichi. On the imbedding problem of spaces of constant curvature in one another. Nat. Sci. Rep. Ochanomizu Univ. 4, 44–50 (1953).

Using Cartan's method of moving frames, the author proves the theorem: Let S_n, $\bar{S}_N$ be spaces of constant curvature K and $\bar{K}$ respectively. Then if either (A) $K>\bar{K}$ and $N=n+1$ or (B) $K=0$, $\bar{K}>0$, and $N=2n-1$ is true, S_n can be imbedded isometrically in $\bar{S}_N$. These results, as well as a generalization of (B) have also been obtained by Liber [C. R. Doklady Acad. Sci. URSS (N.S.) 55, 291–293 (1947); these Rev. 8, 603] whose paper was not accessible to the author. A generalization of (A) was proved by the reviewer in a different way [Proc. Nat. Acad. Sci. U. S. A. 24, 30–34 (1938)]. It should be noted that the definition used by the author is the negative of the usual one which is employed in the present review. *A. Fialkow* (Brooklyn, N. Y.).

Pan, T. K. On a generalization of the first curvature of a curve in a hypersurface of a Riemannian space. Canadian J. Math. 6, 210–216 (1954).

It is well known that the theory of curvature of a curve in V_n can be generalized by taking an arbitrary unit vector along the curve instead of the tangent unit vector. There is another generalization for a curve in a V_n in V_{n+1}. Instead of the direction normal to V_n a general direction is taken, orthogonal to the curve. This new direction can be derived from some congruence of unit vectors or some congruence of hypercones of unit vectors. The concept of first curvature can then be generalized and a theory of curvature can be built on these cones analogous to the ordinary theory.

J. A. Schouten (Epe).

Suguri, Tsuneo. On deformed Riemannian spaces. Mem. Fac. Sci. Kyūsyū Univ. A. 8, 43–55 (1953).

L'auteur étudie la stabilité de propriétés variées d'un espace de Riemann V_n lorsqu'on le déforme, c'est-à-dire quand on ajoute au tenseur métrique g_{ij} sa différentielle de Lie relativement à une certaine transformation infinitésimale. On sait en particulier que la dérivation de Lie commute avec la dérivation covariante. Dans une première partie, l'auteur étudie les formes harmoniques sur V_n, retrouvant des résultats de K. Yano [Ann. of Math. (2) 55, 38–45 (1952); ces Rev. 13, 689] (l'énoncé du théorème 3 du présent travail ne semble pas tout à fait correct au rapporteur). Dans la seconde partie, il montre que le fait pour V_n d'être à courbure constante, espace d'Einstein, espace symétrique, etc., est stable par déformation. Il en est de même pour les propriétés du groupe d'holonomie étudiées par Yano et Sasaki [Proc. Japan Acad. 24, no. 7–8, 7–13 (1948); ces Rev. 14, 87]. *A. Lichnerowicz.*

Lemoine, Simone. Rigidité des V_{n-1} d'un espace riemannien S_n, à courbure constante. C. R. Acad. Sci. Paris 238, 559–561 (1954).

Let S_n be a space of constant curvature K, where $K\neq 0$, and let V_{n-1} be a hypersurface of S_n. It is proved that, if $n>4$, V_{n-1} is rigidly imbedded in S_n. On the other hand, in S_4 there exist hypersurfaces V_3 which are deformable. Such surfaces V_3 are characterized. *C. B. Allendoerfer.*

Mutō, Yosio. On some properties of a fibred Riemannian manifold. Sci. Rep. Yokohama Nat. Univ. Sect. I. 1, 1–14 (1952).

This paper deals with a Riemannian V^{n+m} which is fibred in the usual sense. The local coordinates of a point of V^{n+m} are $(y^1, \cdots, y^n, x, \cdots, x^m)$. The fibres are defined locally as the sets of points Y^n each of which has the same x-coordinates. By a suitable restriction on the allowable coordinate neighborhoods of V^{n+m} the fibres are defined in the large. The base space X is obtained by identifying the points of V^{n+m} which have the same x-coordinates. It is assumed that V^{n+m} has the Riemannian metric:

$$ds^2 = g_{jh}dy^jdy^h + 2g_{j\alpha}dy^jdx^\alpha + g_{\alpha\beta}dx^\alpha dx^\beta,$$

where all the g's are functions of the x's and the y's. This metric induces a correspondence $(y, x)\rightarrow(y+dy, x+dx)$ between the fibre $Y^n(x)$ and the fibre $Y^n(x+dx)$ by the requirement that the vector (dy^i, dx^α) in V^{n+m} be normal to $Y^n(x)$ at the point (y^i, x^α). When this situation holds, $dy^i + \Gamma_\alpha{}^i(y, x)dx^\alpha = 0$ where $\Gamma_\alpha{}^ig_{ij} = g_{j\alpha}$.

The fibres are called isometric when this correspondence is isometric. They are called holonomic if there exists a system of ∞^n submanifolds orthogonal to Y^n. Conditions for these types of correspondence are developed. It is proved that the fibres are isometric if and only if they are geodesic subspaces. The fibres are called parallel if pairs of them are at a constant distance. If the fibres are both isometric and parallel, V^{n+m} is locally a Riemannian product V^nV^m. If the fibres are parallel, the base space X is a Riemannian manifold. In this case a group of holonomy can be defined; its properties are developed. The results of the paper are applied to the 3-sphere, S^3. *C. B. Allendoerfer.*

Mutō, Yosio. Some properties of geodesics in the large in a two-dimensional Riemannian manifold with positive curvature. Sci. Rep. Yokohama Nat. Univ. Sect. I. 2, 1–12 (1953).

An n-dimensional complete positive-definite Riemannian manifold M^n is said to satisfy Hypothesis A if it sectional curvature $K(P, \gamma)$ satisfies the inequality

$$0<L<K(P, \gamma)<H.$$

It is known that such a manifold is compact and that if $L/H\geq h$ (a number near $\frac{3}{4}$) the universal covering space of M^n is homeomorphic to an n-sphere. A number of theorems are presented assuming Hypothesis A, mostly without proof. Some of these are: If M^n has a geodesic loop of length $\leq 2l$

or has a pair of points which can be joined by two (or more) geodesic arcs of length $\leqq l$ (where $l \leqq \pi/\sqrt{H}$), then M^n has a closed geodesic of length $\leqq 2l$. If n is even and M^n is orientable, M^n has no pair of points which can be joined by two geodesic arcs of length $\leqq \pi/\sqrt{H}$.

If M^n satisfies A and is homeomorphic to a two-sphere (hypothesis B), any geodesic loop has arc length $>2\pi/\sqrt{H}$. If, moreover, $2\sqrt{L} \geqq \sqrt{H}$, any geodesic loop cannot be the shortest loop among loops near the geodesic loop in the fixed-end-point problem. When M^n satisfies A and B, the vertical angle of a simple loop satisfies a certain inequality, and the arc length l_0 of a simple closed geodesic satisfies the inequality: $l_0 < (2\pi/\sqrt{H})(H/L)$. *C. B. Allendoerfer.*

Tomonaga, Yasuro. Note on Betti numbers of Riemannian manifolds. I. J. Math. Soc. Japan 5, 59–64 (1953).

L'auteur indique diverses applications d'un théorème de Bochner [Bochner, Ann. of Math. (2) 49, 379–390 (1948); Lichnerowicz, C. R. Acad. Sci. Paris 226, 1678–1680 (1948); ces Rev. 9, 618] reliant le tenseur de courbure et les nombres de Betti d'une variété riemannienne close, orientable, V_n. Si

$$K = R/n - (R_{ij}R^{ij} - R^2/n)^{1/2}$$

est non négatif, toute 1-forme harmonique est à dérivée covariante nulle; si K est strictement positif, $b_1(V_n)=0$. Ce résultat est étendu aux p-formes harmoniques ($p \geqq 2$) en faisant intervenir un scalaire construit à l'aide du tenseur de courbure. Différentes formes de ce résultat sont données. *A. Lichnerowicz* (Paris).

Tomonaga, Yasuro. Note on Betti numbers of Riemannian manifolds. II. J. Math. Soc. Japan 5, 65–69 (1953).

Soit A_{ij} un tenseur symétrique défini positif. Si la forme quadratique

$$Q(X) = (A_{ij}R^i{}_k - \tfrac{1}{2}\Delta A_{ij})X^jX^k$$

est partout positive, toute 1-forme harmonique est à dérivée positive $b_1(V_n)=0$. Les mêmes conclusions sont valables sans l'hypothèse qu'il existe sur V_n un scalaire p^2 tel que $\Delta p^2/2p^2 \leqq K$ (voir reférat précédent; on prend $A_{ij} = p^2 g_{ij}$). Ces résultats sont aussi étendus à des p-formes harmoniques ($p \geqq 2$). *A. Lichnerowicz* (Paris).

Varga, O. Eine Charakterisierung der Finslerschen Räume mit absolutem Parallelismus der Linienelemente. Arch. Math. 5, 128–131 (1954).

A Finsler space F_n is said to satisfy the hypothesis (1) of the unrestricted existence of extremal-fields if it contains a subregion in which there exists a family of extremal-fields whose tangent vectors assume all possible directions at each point. It satisfies hypothesis (2) if the extremal fields of the osculating Riemannian space possess a group of translations whose paths are extremals. It is proved that hypotheses (1) and (2) are necessary and sufficient for F_n (or a subregion of it) to admit absolute parallelism of its line-element. *C. B. Allendoerfer* (Seattle, Wash.).

***Reeb, Georges. Sur les espaces de Finsler et les espaces de Cartan.** Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 35–40. Centre National de la Recherche Scientifique, Paris, 1953.

The author describes a method of developing the theory of Finsler and Cartan spaces as free of coordinates as possible (this is useful for global considerations). Let V_n be a manifold, and let $V_{n,1}$ be the manifold of all covariant vectors of V_n. If π denotes the natural projection of $V_{n,1}$ onto V_n, there is defined intrinsically a 1-form θ_1 on $V_{n,1}$, which at the point (x, ω) [$x \in V_n$, ω a covariant vector at x] is given by $\pi^*\omega$. If x_i are coordinates in V_n, and p_i are the components of covariant vectors, then θ_1 is given by $p_i dx_i$. The 2-form $\Omega_2 = d\theta_1$ ($= dp_i \wedge dx_i$) is a form of maximal rank on $V_{n,1}$. A Finsler space is obtained by considering a $(2n-1)$-submanifold of $V_{n,1}$ with additional properties [e.g. the intersection with the cotangent space at any point of x, the figuratrix, is to be convex]. The characteristic system of the form induced by Ω_2 is a system of ordinary differential equations; its trajectories project into the Finsler geodesics. A Cartan space is obtained similarly by considering a suitable submanifold of the manifold $V_{n,n-1}$ of covariant (exterior) $(n-1)$-vectors over V_n. Several concepts like curvature-vector and -form of a curve, and mean curvature form are developed, and several theorems, concerning, e.g., non-existence of transversal manifolds to the characteristic system of Ω_2, are given. *H. Samelson.*

***Ehresmann, Charles. Introduction à la théorie des structures infinitésimales et des pseudogroupes de Lie.** Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 11, 16 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1953.

This is an exposition of the author's theory of infinitesimal structures [presented briefly in C. R. Acad. Sci. Paris 233, 598–600, 777–779, 1081–1083 (1951); 234, 1028–1030, 1424–1425 (1952); these Rev. 13, 386, 584, 780, 870]. It is concerned with contact elements of higher order, with the general notion of a system of partial differential equations in a manifold, with the idea of germ of a structure of any kind, such as imbedded manifold, "extracted" manifold, leaved structure (Reeb), complex structure, structure associated with any pseudo group of transformation, etc. The central concept is the local jet: Let E, E' be two spaces, x a point of E; maps of open sets of E into E' define the same jet at x if they coincide in some neighborhood of x. Any such map defines a jet at each point of its domain of definition; these sets of jets generate a topology in the set of all jets. Taking E and E' as Euclidean spaces (or C^∞-manifolds), two maps, of class C^r, define the same infinitesimal r-jet at x, if at x their partial derivatives of order $\leqq r$ are equal. r-jets from Euclidean space to a manifold, resp. conversely, are called r-velocities, resp. r-covelocities; they generalize tangent vectors and differentials. Contact- and envelope-elements are defined. Algebraic properties, such as composition of jets, and related groups and groupoids are investigated. Of particular importance is the group L_n^r, consisting of transformations given by n polynomials of degree $\leqq r$ in n variables (with non-vanishing determinant of the linear part); the group operation is substitution, followed by suppression of all terms of degree $>r$. *H. Samelson.*

***Weil, André. Théorie des points proches sur les variétés différentiables.** Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 111–117. Centre National de la Recherche Scientifique, Paris, 1953.

A local algebra A is an associative, commutative algebra of finite dimension over the reals R, with unit and with an ideal I of height m ($I^m \neq 0$, $I^{m+1}=0$), such that $A/I \approx R$; the image in R of $x \in A$ is called the finite part of x; additively one can write $A = R \oplus I$. The standard example is obtained by taking formal power series (in n variables) modulo the $(m+1)$th power of the ideal of series with vanishing con-

stant term. There are algebraic operations on local algebras, such as taking tensor products. Let V be a C^∞-manifold, $D(V)$ the algebra of C^∞-functions on V, x a point of V, A a local algebra. A neighboring point of x of type A is, by definition, a homomorphism of $D(V)$ into A such that the finite part of the image of f is $f(x)$. If A consists of the dual numbers ($a+b\tau$, with $\tau^2=0$), this reduces to the standard definition of a tangent vector at x. Operations on such points, behavior under mappings, etc. are studied. All points of a given type over V form a manifold which is a fiber space over V (generalizing the space of tangent vectors), the prolongement of type A. Their formal properties are studied. A basic theorem says that the prolongement of type A of the prolongement of type B is the prolongement of type $A\otimes B$. An infinitesimal transformation of type A is a cross-section of the A-prolongement. Infinitesimal transformations of type A can be made to act on the Ehresmann-prolongements of V, generalizing the customary action of a vector-field on tensor fields. Numerous applications of these ideas are promised. *H. Samelson.*

NUMERICAL AND GRAPHICAL METHODS

Have

★Walther, A., und Unger, H. Mathematische Zahlentafeln, numerische Untersuchung spezieller Funktionen. Naturforschung und Medizin in Deutschland, 1939–1946, Band 3. Angewandte Mathematik, Teil I, pp. 167–183. Verlag Chemie, Weinheim, 1953. DM 20.00.

This report covers both numerical tables, and analytical investigations of interest for numerical work with special functions. The report consists of two parts: the first of these gives a survey of the activities in this field in Germany in 1939–1946, and the second, a bibliography of some eighty items, with brief indications of the contents in the case of papers which have not been published. Each of these two parts is divided in four sections: (1) elementary functions; (2) tables used in astronomy and geodesy; (3) Bessel functions; and (4) other higher transcendental functions.
A. Erdélyi (Pasadena, Calif.).

Have

★Flügge, W. Four-place tables of transcendental functions. McGraw-Hill Book Co., Inc., New York, N. Y.; Pergamon Press Ltd., London, 1954. 136 pp. $5.00.

These interesting and useful tables give four-figure values of the following functions: $\sin x$, $\cos x$, $\tan x$, $\cot x$ for $x=0(0°.1)90°$; $\sin x$, $\cos x$, $\tan x$, $\sinh x$, $\cosh x$, e^x, e^{-x}, $\ln x$, $J_0(x)$, $J_1(x)$, $Y_0(x)$, $Y_1(x)$, $I_0(x)$, $I_1(x)$, $K_0(x)$, $K_1(x)$, $\mathrm{ber}\, x$, $\mathrm{bei}\, x$, $\mathrm{ber}'\, x$, $\mathrm{bei}'\, x$, $\mathrm{ker}\, x$, $\mathrm{kei}\, x$, $\mathrm{ker}'\, x$, $\mathrm{kei}'\, x$, $\mathrm{erf}\, x$, the Fresnel integrals $C(x)$, $S(x)$, and $\mathrm{Ei}(-x)$, $\mathrm{Ei}\, x$, $\mathrm{Ci}\, x$, $\mathrm{Si}\, x$, all for $x=0(0.01)10.09$;

$$\begin{array}{ll} \tanh x, & x=0(0.01)5.09, \\ e^x-1,\ 1-e^{-x}, & x=0(0.001)0.2009, \\ 1-\mathrm{erf}\, x, & x=1(0.01)3.09, \\ \Gamma(x), & x=1(0.001)2.009, \end{array}$$

elliptic integrals $F(x, y)$, $E(x, y)$ for

$$x=0(2°)90°, \quad y=0(1°)90°.$$

There are also well chosen lists of formulae relevant to the functions tabulated, and short bibliographies of more extensive tables.

The collection invites comparison with the well-known tables of Jahnke and Emde [Funktionentafeln mit Formeln und Kurven, 3. Aufl., Teubner, Leipzig-Berlin, 1938]. The printing and layout of the present collection are much superior; although the total absence of diagrams is noticeable. The accuracy is almost certainly better, at any rate when compared with earlier editions of Jahnke and Emde, because of the better sources now available. The price is similar. In content, however, Jahnke and Emde contains a very much wider range of functions. *J. C. P. Miller.*

Have

★Tables of Lagrangian coefficients for sexagesimal interpolation. National Bureau of Standards Applied Mathematics Series No. 35. U. S. Government Printing Office, Washington, D. C., 1954. ix+157 pp. $2.00.

These tables give coefficients for use in applying Lagrange's interpolation formula $f(x)=\sum A_r(x)f_r$, where f_r are equally spaced values from a function table, n in number for an 'n-point formula'. The values are assumed to be at unit intervals, the unit being one, such as a degree or an hour, which it is customary to divide into 60 minutes, each, in turn, divisible into 60 seconds. The tables give the coefficients A_r for the 3-, 4-, 5- and 6-point formulae for x varying from 0 to unity at intervals of 1 second. There are thus 3600 arguments, expressed in minutes and seconds, in each case, although with the 4- and 6-point formulae symmetry allows the tables to be condensed so that one line of coefficients has two arguments, θ and $1-\theta$.

All values are given to 8 decimals, rounded to the nearest final unit. This means that $\sum A_r(x)$ is not always precisely equal to unity, as it should be; the correct sum is, in this case, much more important than correct rounding off, since the inaccuracy of the sum limits the use of the tables to the interpolation of function tables with 8 figure values whereas a correct unit sum would allow accurate use with the wider class of tables having 8 figure first differences. With care, an appropriate adjustment may readily be made.

It is to be hoped that these tables will not encourage the use of sexagesimal arguments in trigonometric and other tables. *J. C. P. Miller* (Cambridge, England).

Have

★Table of secants and cosecants to nine significant figures at hundredths of a degree. National Bureau of Standards Applied Mathematics Series No. 40. U. S. Government Printing Office, Washington, D. C., 1954. vi+46 pp. 35 cents.

This useful table gives $\sec x$ and $\cos x$ to 9 significant figures for $x=0°(0°.01)90°$ and is arranged semiquadrantally. It forms a comparison volume to no. 5 (1949) of the same series [these Rev. **10**, 740]. *J. C. P. Miller.*

Wilkes, M. V. A table of Chapman's grazing incidence integral $\mathrm{Ch}(x, \chi)$. Proc. Phys. Soc. Sect. B. **67**, 304–308 (1954).

This table gives

$$\mathrm{Ch}(x, \chi)=x \sin \chi \int_0^\chi \exp\,\{x-x \sin \chi/\sin \lambda\}\, \mathrm{cosec}^2\, \lambda d\lambda$$

to 3D for $x=50(50)500(100)1000$, $\chi=20°(1°)100°$, excluding values of $\mathrm{Ch}(x, \chi)$ which exceed 100. The table was prepared on EDSAC, using a Gauss 5-point-formula repeatedly, with automatic adjustment of the length of the interval. A more

complete discussion of the machine techniques used is to appear elsewhere. Various checks were applied: recalculation for $\chi=20°, 21°, 70°$, differencing in both directions, and use of the relation

$$\mathrm{Ch}(x,\chi)+\mathrm{Ch}(x,\pi-\chi)=2\exp\{x-x\sin\chi\}\,\mathrm{Ch}(x\sin\chi,\tfrac{1}{2}\pi).$$

The values are expected to be correct to within a unit in the last place. *J. Todd* (Washington, D. C.).

Dunski, Ch. Vital. Les fonctions de Bessel d'argument complexe $x\sqrt{j}$ et les fonctions de Kelvin d'ordre zéro et 1. Bull. Soc. Roy. Sci. Liège **23**, 52–59 (1954).

This paper contains two pages of tables of ber x, bei x, ker x, kei x, $\mathrm{ber}_1\,x$, $\mathrm{bei}_1\,x$, $\mathrm{ker}_1\,x$, and of the real and imaginary parts of $J_0(xi^{3/2})$, $H_0^{(1)}(xi^{3/2})$, $i^{1/2}J_1(xi^{3/2})$, $i^{1/2}H_1^{(1)}(xi^{3/2})$. There are 25 values of x ranging from 10 to 72, and the values of the functions were computed from the asymptotic expansions. *A. Erdélyi* (Pasadena, Calif.).

Nelšuler, L. Ya. On the tabulation of Taylor series for functions of three variables. Doklady Akad. Nauk SSSR (N.S.) **94**, 797–800 (1954). (Russian)

The author discusses the problem of tabulating a function of $u(x_1, x_2, x_3)$ of three variables together with the nine terms of Taylor's expansion carried as far as to include the second order terms. When the function v is intractable a function $\Phi(t)$ is chosen and the function

$$u(x_1, x_2, x_3)=\Phi[v(x_1, x_2, x_3)]$$

is tabulated in lieu of v. The discussion is only general; no example is given. *D. H. Lehmer* (Berkeley, Calif.).

Householder, Alston S. Generation of errors in digital computation. Bull. Amer. Math. Soc. **60**, 234–247 (1954).

The author considers the practical evaluation, on a digital computer, of $f(x)$, given x: (1) it will be necessary in general to evaluate $f(x)$ for some x^* near x; (2) it will be necessary in general to evaluate f_a, a rational approximation to f rather than f itself; (3) instead of evaluating f_a, we may have to be content with some f^*. The approximations introduced in (1), (3) are necessary in view of the digital computation contemplated. He calls $f(x)-f(x^*)$ the propagated error, $f(x^*)-f_a(x^*)$ the residual error, and $f_a(x^*)-f^*(x^*)$ the generated error: it is with the last that he is concerned.

The dependence of the generated error on the detailed behavior of the computer and on the detailed arrangement of the computation is illustrated by a discussion of the evaluation of $a/(bc)$. First by the sequence: $b\times c$, $a/(b\times c)$; second by a/b, $(a/b)/c$; and third by this last process modified to take account of the fact that in the computer under discussion (ORACLE) the remainder in division is (fortunately) accessible. Conditions which are sufficient to ensure the first being more efficient than the second are given; in the event that the values of a, b, c, are not known in advance, it is easy to program the machine to carry out the evaluation by the first process, and if this is not the better, to recompute using the second method.

The author points out that in many cases the errors possible are unsymmetrical: for instance, in a division, the machine may produce the positive remainder not the one of least absolute value. He therefore introduces the concept of uncertainty, η: if it is known that the true f and f^*, the reputed f, satisfy $f^*-\delta_l\leq f\leq f^*+\delta_u$ (where the δ_l, δ_u need not both be positive), then $\eta=\delta_l+\delta_u$. [Cf. J. Kuntzmann, Ann. Inst. Fourier Grenoble **2**, 197–205 (1951); these Rev. **13**, 162.] The evaluation of a polynomial of degree n by a direct method and by the backward recurrence of Horner (synthetic division) are compared in terms of the resulting uncertainties: it is found that the Horner method is better by a factor of n and that a further improvement can be had if the coefficients can be obtained inductively. The solution of equations $f(x)=0$, in particular $x^2-a=0$, is discussed in terms of uncertainty. *J. Todd* (Washington, D. C.).

Shanks, Daniel. A logarithm algorithm. Math. Tables and Other Aids to Computation **8**, 60–64 (1954).

Gagua, M. E. On the computation of the values of Bessel functions of the first kind. Soobščeniya Akad. Nauk Gruzin. SSR **14**, 455–458 (1953). (Russian)

For the numerical evaluation of Bessel functions with complex order and variable the author considers the use of Sonine's integral representation

$$J_\nu(z)=\frac{(z/2)^\nu}{2\pi i}\int_{-\infty}^{(0+)}\exp\left(t-\frac{z^2}{4t}\right)t^{-\nu-1}dt.$$

The contour taken consists of a circle of radius R, and the segments from $-\infty$ to $-R$ of both sides of the branch-cut along the negative real axis. If R is large enough, the integral along the segments can be neglected; the integral around the circle can be written as a finite integral by introducing the angle as the independent variable, and this integral can be evaluated by Simpson's rule. The author gives estimates for the integrals along the segments, and for the error-term in Simpson's rule. *A. Erdélyi.*

Salzer, Herbert E. New formulas for facilitating osculatory interpolation. J. Research Nat. Bur. Standards **52**, 211–216 (1954).

Hermite's interpolation formula [J. Reine Angew. Math. **84**, 70–79 (1877)] gives the unique polynomial of degree $\sum_{i=1}^n m_i-1$ for which the ordinates and the derivatives up to the order m_i-1 at $x=x_i$ $(i=1, 2, \cdots, n)$ have prescribed values. A demonstration of the existence and uniqueness of this polynomial has been given by Curry [Portugaliae Math. **10**, 135–162 (1951), pp. 142–143; these Rev. **13**, 632]. The author considers the case $m_1=m_2=\cdots=m_n=2$, employing the ordinates and first derivatives. If the x_i are equally spaced, so that $x_i=x_0+ih$, the required polynomial of degree $2n-1$ can be written in the form:

$$f(x_0+ph)=\sum_{i=1}^n(\alpha_i f_i+\beta_i h f_i')\Big/\sum_{i=1}^n\alpha_i,$$

where f_i and f_i' are the given ordinates and first derivatives, $\alpha_i\equiv a_i/(p-i)^2+b_i/(p-i)$, and $\beta_i\equiv a_i/(p-i)$, the a_i and b_i being numerical coefficients depending on n but independent of p. Exact values of the a_i and b_i are given for $n=2$ to 11, inclusive.

f^{2n}

When the formula is used for interpolation of a function $f(x)$, the remainder is $h^{2n}[L^{(n)}(p)]^2f^{(2n)}(\xi)/(2n)!$, where ξ is between the least and the greatest x_i and $L^{(n)}(p)=\prod_{i=1}^n(p-i)$. Approximate upper bounds for $[L^{(n)}(p)]^2/(2n)!$ are given. The author shows that this interpolation formula is very much more accurate than the corresponding n-point Lagrangian formula and considerably more accurate than even the $2n$-point Lagrangian formula at intervals of h. It is therefore useful with tables where the derivative of the function is also tabulated or is easily obtained, such as Bessel and probability functions, and many elementary functions. [This paper perpetuates an inaccurate use of the

term "osculatory" common among actuarial writers (of which the reviewer has also been guilty in the past) to include the case of curves whose first derivatives only are equal at their point of tangency. The reviewer believes the author is in error in referring to the polynomial given by Fort [Finite differences and difference equations in the real domain, Oxford, 1948, p. 85; these Rev. **9**, 514] as of degree $mn-1$, since Fort considers here the general case in which the number of given derivatives is not necessarily the same for the different arguments. Misprint: In the fifth line after formula (3) read f_p for fp.] *T. N. E. Greville.*

Gross, Oliver. Polynomial-like approximation. Math. Tables and Other Aids to Computation **8**, 58–60 (1954).

★Allen, D. N. de G. Relaxation methods. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. ix+257 pp. $7.50.

Two chapters on linear algebraic equations introduce the basic ideas and acceleration devices of the relaxation method of successive approximation, and a third illustrates its application to framework problems, from which the method originated and obtained its vocabulary and notation. Apart from a later chapter on eigen-value problems in algebraic equations, the remainder is concerned with differential equations. These are mostly partial and of elliptic type, but there is a short introductory chapter on ordinary differential equations with two-point boundary conditions. Then follows the treatment of the equations of Laplace and Poisson, the "quasi-plane-potential" equation

$$\frac{\partial}{\partial x}\left(x\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(x\frac{\partial\phi}{\partial y}\right)+\Phi=0,$$

the biharmonic equation and its corresponding pair of simultaneous second-order equations, and problems introduced by multi-connected regions and boundaries not initially known, as for example in streamline flow with a free surface and at the plastic-elastic boundaries in the torsion problem. Eigen-solutions of differential equations, ordinary and partial, are obtained by two different methods. New material is introduced in the last two chapters, which discuss the relaxational treatment of non-elliptic equations, with particular attention to the heat conduction equation, and a method for three-dimensional relaxation with the use of a single diagram.

The author's aim is "to explain as clearly as possible how to relax" and other relevant matters have been deliberately excluded. This object has been achieved quite admirably. The explanation is detailed, sound and well illustrated, and each new topic is introduced at the proper place. The reader is exhorted to try his hand on further illustrative problems.

To the expert relaxer, however, the omission of "other topics such as convergence, finite-difference and truncation errors, parallel procedures used in relaxation and other computational methods, etc.," will be a disappointment. This is surely more than a "philosophy of relaxation". The expert wants to know the circumstances in which relaxation is better or worse than other methods, whether the techniques described are as accurate and as economic as possible and where, in short, relaxation fits into his computational repertoire. The Southwell school of relaxation is now twenty years old, and except for some mention by F. S. Shaw [An introduction to relaxation methods, Dover, New York, 1953; these Rev. **15**, 353], any attempt at such an evaluation is scattered among the literature of scientific papers.

This reviewer, at least, looks forward with impatience to the half-promised volume devoted to these further matters of "great intrinsic interest and importance". If it is as good as the volume under review, he will buy it notwithstanding the fact that he has to pay about twice as much, for an American publication, as the cost of a publication in England of the same size and quality. *L. Fox.*

Dantzig, George B., and Orchard-Hays, Wm. The product form for the inverse in the simplex method. Math. Tables and Other Aids to Computation **8**, 64–67 (1954).

Èĭtkin [Aitken], A. On the factorization of polynomials by iterative methods. Uspehi Matem. Nauk (N.S.) **8**, no. 6(58), 71–86 (1953). (Russian)

Translation by A. Ya. Belostockiĭ of Aitken's paper in Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 174–191 (1951); these Rev. **12**, 860.

Belostockiĭ, A. Ya. On a method of solution of algebraic equations. Uspehi Matem. Nauk (N.S.) **8**, no. 6(58), 87–96 (1953). (Russian)

This paper is published along with a Russian translation [see the paper reviewed above] of the first of two papers by Aitken on the iterative factorization of polynomials [Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 174–191 (1951); cf. also ibid. **63**, 326–335 (1952); these Rev. **12**, 860; **14**, 209]. Given a polynomial $f(x)$ of degree n and a divisor of degree $m<n$, each with leading coefficient unity, Aitken considers the penultimate remainder after dividing f by d_0 whose coefficients differ only slightly from those of d. Let ϵ_0 designate the vector of deviations of the coefficients, and let ϵ_1 designate the deviation of the coefficients of the reduced penultimate remainder, whose leading coefficient is also unity. Aitken exhibits the matrix R such that $\epsilon_1=R\epsilon_0+o(\epsilon_0)$, and finds its latent roots. The present author shows that one can determine a polynomial k of degree m, associated with a given f, such that the R arising from the division of kf by d_0 will vanish, thus yielding an iteration of higher order. *A. S. Householder* (Oak Ridge, Tenn.).

Barna, Béla. Über die Divergenzpunkte des Newtonschen Verfahrens zur Bestimmung von Wurzeln algebraischer Gleichungen. I. Publ. Math. Debrecen **3** (1953), 109–118 (1954).

This is a continuation of Rényi's [Mat. Lapok **1**, 278–293 (1950); these Rev. **12**, 321] and the author's [Publ. Math. Debrecen **2**, 50–63 (1951); these Rev. **13**, 18] earlier work on the convergence of Newton's method of iteration. In the present paper the author considers real polynomials of degree $m\geqq 4$ (his previous paper was restricted to the case $m=4$) which have m distinct real roots. He discusses and classifies the points of divergence of the iteration and shows that they form a non-enumerable set. *E. Lukacs.*

Krylov, V. I. On computation of an indefinite integral with a small number of values of the integrated function. Doklady Akad. Nauk SSSR (N.S.) **94**, 613–614 (1954). (Russian)

It is required to evaluate an integral

$$y(x)=y_0+\int_{x_0}^{x} f(t)dt$$

at each of the equally spaced points $x_n=x_0+nh$. The author proposes to select points $\alpha+p_ih, \cdots, \lambda+l_ih$ for an optimal

representation

$$\int_{x_n}^{x_n+h} f(t)dt \doteq A_0 f(\alpha)+\sum_1^a A_i f(\alpha+p_i h)+\cdots$$
$$+L_0 f(\lambda)+\sum_1^l L_i f(\lambda+t_i h).$$

It is assumed that x_n is not too close to either end of the range. Three theorems are stated relating to the degree of polynomial $f(x)$ for which the representation would be exact. The degree is said to be $n+m$ where m is the number of points $\alpha, \cdots, \lambda$, and where $n+1=a+\cdots+l+m$. Presumably this n is not the same as the index on x_n.

A. S. Householder (Oak Ridge, Tenn.).

Ginzburg, B. L. Formulas for numerical quadrature most convenient for application. Uspehi Matem. Nauk (N.S.) 9, no. 2(60), 137–142 (1954). (Russian)

The "degree of precision" of a quadrature formula

$$\int_{-1}^{1} f(x)dx = 2\sum_1^n A_i f(x_i)+R_p$$

is the degree p of the polynomial $f(x)$ of lowest degree for which the remainder R_p does not vanish identically. After remarking about the inconvenience of the Gaussian formula, the author asks for the separation h of the equally spaced, symmetrically placed abscissae $x_1, \cdots, x_n$ for which the degree of precision is as great as possible. For $n=2m$ or $2m+1$ an equation of degree m must be solved, and it turns out that there is one and only one solution for which all points x_i will lie within the interval $(-1, 1)$. For these "optimal formulas", the author tabulates, for $n \leqq 11$, the h and the A_i, and compares the error with the error in the Gaussian formula of equal precision. *A. S. Householder.*

Todd, John. Evaluation of the exponential integral for large complex arguments. J. Research Nat. Bur. Standards 52, 313–317 (1954).

The integral $E_1(z)=\int_z^\infty e^{-u}u^{-1}du$, $z=x+iy$, is considered for z not being located on the negative real axis and for the path of integration $u=z+r$ with r real and $r \geqq 0$. The emphasis is on large values of z as $|z| \geqq 10$. Two methods of evaluation are compared with one another:

$$(1) \qquad E_1(z)=e^{-z}\left(\frac{1}{z}-\frac{1!}{z^2}+\frac{2!}{z^3}-\cdots-\frac{(n-1)!}{(-z)^n}\right)+R_n,$$

$$R_n=(-1)^n n!\int_z^\infty e^{-u}u^{-n-1}du$$

$$(2) \qquad E_1(z)=e^{-z}\sum_i \frac{\lambda_i^{(n)}}{z+x_i^{(n)}}+R_n^*$$

with the numbers $x_i^{(n)}$ as zeros of the Laguerre polynomial $L_n(t)$ and with the coefficients $\lambda_i^{(n)}$ as corresponding Christoffel numbers. The evaluation proceeds by neglecting the remainders R_n and R_n^*. In both cases n is chosen near to $|z|$ for $x \geqq 0$ and near to $|y|$ for $x<0$. It turns out that $|R_n^*|$ is of the order of the square of $|R_n|$, which makes the Laguerre quadrature method preferable. Some numerical examples and remarks on other methods conclude the paper.

H. Bückner (Schenectady, N. Y.).

Saul'ev, V. K. On finding eigenvalues by the method of grids. Doklady Akad. Nauk SSSR (N.S.) 94, 1003–1006 (1954). (Russian)

For the m-dimensional, self-adjoint boundary-value problem, bounds for $|\lambda_p-\lambda_p^{(h)}|$ are obtained, where $\lambda_p^{(h)}$ represents the pth eigenvalue for the finite-difference equations and h is a measure of the mesh size. *A. S. Householder.*

Ricaldoni, J., and Ponce, A. A new method of solution of the Laplace equation. Bol. Fac. Ingen. Montevideo 5, 3–15 (1954). (Spanish)

The Laplacian equation $\partial^2 z/\partial x^2+\partial^2 z/\partial y^2=0$ for a given domain and with z prescribed at the boundary C is approximately solved by means of a physical model representing a set of difference equations. With respect to a cartesian (x, y, z)-coordinate system and to a mesh $x=ih$, $y=kh$ (i, k integers, h width of step) of the (x, y)-plane two systems of cords $z=f_k(x)$, $y=kh$ and $z=g_i(y)$, $x=ih$ are established. Proper twisting of the two systems results in $f_k(ih)=g_i(kh)$. The cords are passed over stationary pulleys distributed along C and their free ends are loaded by individual weights. Each pulley has a height z equal to the prescribed boundary value. The x-component of tension of cord $z=f_k(x)$, $y=kh$ has a constant value C_k, and the y-component of tension of cord $z=g_i(y)$, $x=ih$ has a constant value D_i. The equilibrium condition at the intersection of the two cords is

$$C_k\frac{\partial f_k}{\partial x}\bigg|_{ih-0}^{ih+0}+D_i\frac{\partial g_i}{\partial y}\bigg|_{kh-0}^{kh+0}=0;$$

this represents the difference method approximation to the problem if the constants C_k and D_i have the same value for all combinations i, k. This condition is satisfied by adjusting the weights at the free ends of the cords by trial and error. The paper contains some photographs of the model and numerical results of a problem of plane elasticity with respect to the dilatation. The numerical data are sufficiently close to the exact solution. *H. Bückner.*

Mitchell, A. R. Round-off errors in relaxational solutions of Poisson's equation. Appl. Sci. Research B. 3, 456–464 (1954).

The author discusses the numerical solution of the following differential equations:

$$\frac{d^2\phi}{dx^2}+k\phi=F(x), \qquad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=F(x, y).$$

The second derivatives are replaced by second central differences and the resulting difference equations are satisfied (to within a round-off error) at every point of a mesh covering the region of integration. What is the magnitude of the round-off error ϵ in ϕ at each mesh-point? The author gives various formulas for ϵ; he studies the behavior of ϵ as the number of mesh-points approaches infinity, and (in the two-dimensional case) as the mesh-ratio $K=\Delta y/\Delta x$ is varied. These formulas will be useful, but should be employed with care, since not all possible round-off errors are considered and the dependence of ϵ on K appears erroneous (p. 463). The reviewer feels that the "separation of variables method" [see M. A. Hyman, Appl. Sci. Research B. 2, 325–351 (1952); these Rev. 13, 993], used here for the two-dimensional case and underlying the one-dimensional case, allows an analytically simple approach to general error studies for an important class of partial difference problems.

M. A. Hyman (Washington, D. C.).

Sternberg, R. L., and Kaufman, H. A general solution of the two-frequency modulation product problem. I. J. Math. Physics 32, 233–242 (1954).

Let $Y(x)$ be a given real function. Then, if P and Q are constants, $Y(P\cos u+Q\cos v)$ can be expanded in a double

Fourier series. The physical problem which motivated this study is concerned with such a series evaluated for $u=pt+r$ and $v=qt+s$ where p, q, r, and s are constants and t is the time. The first step is to approximate Y as a linear combination of "ramp functions" of the form $(x-h)+|x-h|$. Recursion relations are found for the Fourier coefficients of the ramp functions. It results that the coefficients for Y can be approximated as linear combinations of four functions introduced by W. R. Bennett [Bell System Tech. J. **26**, 139–169 (1947)]. *R. J. Duffin* (Pittsburgh, Pa.).

Sternberg, Robert L. A general solution of the two-frequency modulation product problem. II. Tables of the functions $A_{mn}(h, k)$. J. Math. Physics **33**, 68–79 (1954).

This paper tabulates the four functions of W. R. Bennett discussed in the above review. They are defined by

$$A_{mn}(h, k)=(2/\pi)^2\int\int(\cos u+k\cos v-h)\cos mu\cos nv\, dudv$$

for $m, n=0, 1$; $|h|\leqq 2$, and $0<k\leqq 1$. The region of integration is defined by the relations $\cos u+k\cos v\geqq h$, $0\leqq u$, and $v\leqq\pi$. The tables are given to six decimal places. The values chosen for k are .001, .01, .1 and 1. Some tables for higher values of m and n are included; these can also be found by recursion identities. *R. J. Duffin* (Pittsburgh, Pa.).

***Walther, A., und Dreyer, H.-J. Mathematische Maschinen und Instrumente. Instrumentelle Verfahren.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 3. Angewandte Mathematik, Teil I, pp. 129–165. Verlag Chemie, Weinheim, 1953. DM 20.00.

This is a review of German developments in mathematical machines, both digital and continuous, written as a sequence of references to published articles. Included in the discussion of digital machines are a number of developments in desk calculators, with emphasis on double or linked machines, their applications, accounting machines, and punched card devices. The section on automatic calculators describes various navigational devices with built-in function tables and two relay calculators. The section on continuous devices refers to drawing board instruments, harmonic analyzers, differential analyzers, polynomial and linear system equation solvers, and electrolytic tanks. *F. J. Murray.*

Hayashi, Shigenori. Applications of the Laplace transformation and electronic analog computers in the study and design of automatic control systems. Bull. Eng. Res. Inst. Kyoto Univ. **5**, 1–10 (1954). (Japanese. English summary)

Expository paper.

Brock, Paul, and Rock, Sibyl. Problems in acceptance testing of digital computers. J. Assoc. Computing Mach. **1**, 82–87 (1954).

Burks, Arthur W., Warren, Don W., and Wright, Jesse B. An analysis of a logical machine using parenthesis-free notation. Math. Tables and Other Aids to Computation **8**, 53–57 (1 plate) (1954).

Leiner, Alan L. System specifications for the DYSEAC. J. Assoc. Computing Mach. **1**, 57–81 (1954).

Löfgren, L. E. Partial drift compensation in electronic d-c analog computers for differential equations. Appl. Sci. Research B. **4**, 109–123 (1954).

Azaroff, Leonid V. A one-dimensional Fourier analog computer. Rev. Sci. Instruments **25**, 471–477 (1954).

Grant, Fraser S. A theory for the regional correction of potential field data. Geophysics **19**, 23–45 (1954).

The author computes the value $\gamma(x, y)$ of the regional anomaly at each point $P(x, y)$ of a given map by smoothing the observed quantity $g(x, y)$ with the aid of Cooper's method so that, in principle, the regional anomaly is given by

$$(*)\qquad \pi\gamma(x, y)=\beta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(u, v)\exp[-\beta s^2]dudv,$$

where $s^2=(x-u)^2+(y-v)^2$. Two very important objections to the use of this method, namely the impossibility of integration over the whole plane because of the overlapping of local anomalies and the arbitrariness in the choice of smoothing parameter β, are completely eliminated in this contribution to applied geophysics. The value of β to be used in a particular case is determined from the map itself with the aid of a special condition applied at the point where the maximum of the local anomaly is located. The expression (*) of $\gamma(x, y)$ is transformed in such a way that the double integration over the whole plane is reduced to a simple integration over a finite interval. Valuable examples and tables annexed to this article illustrate and facilitate the applications. *E. Kogbetliantz.*

Pentkovskiĭ, M. V. Approximate nomograms from aligned points with two parallel scales. Doklady Akad. Nauk SSSR (N.S.) **83**, 27–30 (1952). (Russian)

It is assumed that u and w lie between known extremes and that v, to be determined so that $F(u, v, w)=0$, has a known characteristic [Pentkovskiĭ, same Doklady (N.S.) **66**, 339–342 (1949); these Rev. **11**, 406] determined by the precision with which v is to be found. From this a rectilinear scale for v is determined. A scale for u is adopted by projecting the v-scale through a chosen point to a chosen parallel line. Let P denote the intersection of the tangents at the ends of a typical w-arc of the tangential nomogram that could be constructed, let N be the point on this w-arc for which NP is parallel to the v scale and let Q be the midpoint of NP. The approximate nomogram uses the points Q instead of the w-arcs. The construction described is analytically formulated and a bound for the error in using such a nomogram to find v is deduced and discussed, under the assumption that the w-arcs are without inflection points or cusps. *R. Church* (Monterey, Calif.).

Michajlov, Il. Etude stéréométrique de l'exactitude du cubage simple et sectionné des tiges des arbres. Bull. Soc. Math. Phys. Macédoine **4** (1953), 6–19 (1954). (Macedonian. French summary)

Saxov, Svend, and Nygaard, Kurt. Residual anomalies and depth estimation. Geophysics **18**, 913–928 (1953).

The depth estimations studied in this article are based on the approximate expression R of the first vertical derivative of the observed quantity $g(x, y)$ at a point $P(a, b)$ the distance from which is denoted by r:

$$R=-\frac{\bar{g}(r_1)-\bar{g}(r_2)}{r_1-r_2},$$

where $\bar{g}(r)$ denotes the mean value of $g(x, y)$ on the circumference of a circle of radius r and center $P(a, b)$. This

quantity R has nothing in common with the residual anomaly resulting from the subtraction of the regional from the observed anomaly. The various maps of the same quantity R deduced from a given map for different choices of r_1 and r_2 differ too much to be considered as good approximations to the map of the first derivative. In the reviewer's opinion the maps of the first vertical derivative computed directly with the aid of very exact methods (such as for instance V. Baranov's method) are much more reliable and yield a much better picture of the distribution of sources (subterranean masses in the case of gravity) than the maps of R. *E. Kogbetliantz* (New York, N. Y.).

RELATIVITY

✶Törnebohm, Håkan. A logical analysis of the theory of relativity. Almqvist and Wiksell, Stockholm, 1952. 273 pp. Sw. Cr. 25.00; bound Sw. Cr. 30.00.

The author aims "at an overall picture of the theory of relativity in relation to classical pre-relativity physics", rather than at a special examination of fundamental concepts and principles from the standpoint of a particular philosophical position. As such, it is a welcome addition to the existing critiques of the theory. In order to provide the subject matter for this analysis, Törnebohm devotes Part I, consisting of about three-fifths of the book, to an exposition of the special and general theories of relativity, with emphasis on those matters of particular methodological or conceptional import. The remaining Part II contains the logical analysis proper, against a general epistemological background.

Part I opens with a careful exposition of the kinematical aspects of the special theory, emphasizing its relation to the Michelson-Morley experiment and the role of the Lorentz transformations, in terms of the Minkowski geometry of space-time. The transition to the general theory of relativity is essayed by an exposition of the ill-starred problem of the rotating disc, suggesting that the resolution of this problem is to be obtained with the aid of Reichenbach's "philosophical principle of the relativity of geometry"; this, to the reviewer, is quite unsatisfying. The remainder of this and the next chapter present a rapid development of the equations of motion and the gravitational field equations, shunning neither the tensor calculus nor its application to Riemannian geometry. The remaining two chapters of Part II contain a brief account of the spherically symmetric gravitational field and related topics, and of relativistic cosmology (as of 1934).

Three points call for special comment. The first is that initially Törnebohm seems, in Sec. 4 "The two principles in the special theory of relativity", to distinguish clearly between the principle of covariance—a methodological principle imposed upon the mode of formulation of the theory—and the true principle of special relativity—the statement of an empirical situation arising from the inferred or postulated equivalence of observers whose results of measurement are related through the Lorentz transformation group. But in progressing to the situation met in the general theory of relativity (p. 68 and later in Sec. 41 of Part II), this distinction between the formal principle of covariance and the factual principle of the relativity of observers of a preferred class seems to have been lost; the latter could well have been illustrated in terms of the cosmological model. The second point is that the author finds an important distinction between the special and general theories, in that since only the latter takes account in its foundations of the causal relation between the space-time continuum and its material content, only in it can we hope to find a scientific "explanation" of the origin of inertia. But his attempt in Secs. 22, 23 to realize Mach's principle (by means of Thirring's approximate solution of the field of a rotating sphere) is at best only suggestive. Finally, it is to be regretted that he makes only passing mention (p. 106) of the fact that the equations of motion are deducible from the non-linear field equations, as shown by Einstein, Infeld and Hoffmann [Ann. of Math. (2) 39, 65–100 (1938)], for surely this development is of great moment to the logical structure of the theory.

Part II contains the logical analysis proper, from a pragmatic standpoint which explicitly rejects the excesses of extreme operationalism, which render this latter impotent in dealing with atom or cosmos alike. The analytical machinery required is carefully developed from primitive sense data, and the notion of physical things, through successive stages of sophistication to the concepts of "descriptive matrices" and of "functors on the fifth level", in terms of which the final expression of the laws of nature is obtained. The various steps in this intricate process of theory-building are well illustrated by reference to simple observation, to mechanics to electromagnetism, and finally to the relativity theories themselves. *H. P. Robertson.*

García, Godofredo. New methods in A. Einstein's general theory of relativity. Actas. Acad. Ci. Lima 15, 99–135 (1952). (Spanish)

L'auteur donne un nouvel exposé des principes fondamentaux de la relativité générale, dans le cas de distributions énergétiques représentés sous forme de fluide parfait; les notations "absolues" et le repère mobile sont systématiquement utilisées. On trouve dans le travail des formes explicites des équations de champ et des équations des trajectoires matérielles dans le cas où l'on a pu adopter des coordonnées orthogonales. *A. Lichnerowicz* (Paris).

García, Godofredo. On contemporary physics and the Schrödinger equation in the theory of relativity. Actas Acad. Ci. Lima 16, nos. 3–4, 3–55 (1953). (Spanish)

Tharrats Vidal, Jesús M.ª. Foundations of a new unitary field of gravitation and electricity. I. An. Real Soc. Españ. Fís. Quím. Ser. A. Fís. 50, 41–44 (1954). (Spanish. English summary)

Introductory paper of a proposed series, in which the unified field is to be defined locally with the aid of a cubic hypersurface as the absolute. The case in which the hypercubic degenerates into a hyperquadric and a hyperplane is interpreted physically as that in which the unitary field is decomposed into an Einsteinian gravitational field and a Maxwellian electromagnetic field. *H. P. Robertson.*

v. Laue, M. Le Chatelier-Braunsches Prinzip und Relativitätstheorie. Z. Physik 137, 113–116 (1954).

For a body with uniform temperature T and pressure p, the velocity and momentum have the same direction, and the entropy equation reads

$$dU = q dG + T dS - p dV,$$

where U is energy, S entropy, V volume, and q, G the magni-

tudes of velocity and momentum respectively. To this is applied the Le Chatelier-Braun principle which asserts that if $dU=\sum X_i dx_i$, then for a principal quantity (x_h) and intensity (X_h) and a neighbouring quantity (x_n) and intensity (X_n) there hold the inequalities

$$\left(\frac{\partial x_h}{\partial X_h}\right)_{X_n} > \left(\frac{\partial x_h}{\partial X_h}\right)_{x_n},$$

no matter, in the case of an unindicated suffix, whether the quantity or the intensity is held constant. The choice $x_h=G$, $X_h=q$, $x_n=S$, $X_n=T$ gives

$$\left(\frac{\partial G}{\partial q}\right)_T > \left(\frac{\partial G}{\partial q}\right)_S,$$

whether p or V is held constant. Other choices of x_n, X_n lead to other inequalities, and these are combined, leading, for example, to

$$\left(\frac{\partial G}{\partial q}\right)_{p,T} > \left(\frac{\partial G}{\partial q}\right)_{p,S} > \left(\frac{\partial G}{\partial q}\right)_{V,S}.$$

Some of the inequalities are verified by the thermodynamics of special relativity, and the limiting Newtonian case is considered. *J. L. Synge* (Dublin).

Narlai, Hidekazu. On some linear equivalence in kinematic relativity. Sci. Rep. Tôhoku Univ., Ser. 1. **37**, 240–248 (1953).

The author isolates a unique linear equivalence leading to a velocity-distance law which does not depend upon epoch. This law is of the form $V/c=\tanh(\nu R/c)$, where ν is a constant (positive for red-shifts). The relativistic acceleration of fundamental particles is always outward and so cannot be treated by the Newtonian analogy employed by Whitrow and Randall [Monthly Not. Roy. Astr. Soc. 111, 455–467 (1951); these Rev. 13, 786]. The author discusses the relation between the time-scale associated with his equivalence and Milne's t-scale, obtains the formula for local density in the world-model based on his equivalence, and appropriately adopts Milne's treatment of the Keplerian problem (with application to a spiral galaxy model). *G. J. Whitrow* (London).

Rayner, C. B. The application of the Whitehead theory of relativity to non-static, spherically symmetrical systems. Proc. Roy. Soc. London. Ser. A. **222**, 509–526 (1954).

The author shows how Whitehead's theory may be used to find the gravitational field of continuous matter in motion. Formulae in polar coordinates are developed for a general non-static spherically symmetric system, and are then applied to a uniformly expanding world model in which proper mass is conserved. With terms in $d\theta$, $d\phi$ omitted, the line-element of this model reads

$$ds^2=c^2\left\{1-\frac{A(3+\alpha^2)}{ct(1-\alpha^2)^{3/2}}\right\}dt^2 +\frac{8A\alpha c\,dt\,dr}{ct(1-\alpha^2)^{3/2}}-\left\{1+\frac{A(1+3\alpha^2)}{ct(1-\alpha^2)^{3/2}}\right\}dr^2,$$

where $\alpha=r/ct$ and A is a constant involving density. The motions of test particles and photons are discussed, and also red-shifts. An appendix deals with sections of a world-tube by a null cone. *J. L. Synge* (Dublin).

Clark, G. L. The problem of two bodies in Whitehead's theory. Proc. Roy. Soc. Edinburgh. Sect. A. **64**, 49–56 (1954).

The author shows that: according to Whitehead's Relativity Theory, in the two-body problem an acceleration of the center of gravity appears, similar to that mistakenly deduced by Levi-Cività in his paper on Einstein's Relativity Theory [Amer. J. Math. 59, 225–234 (1937)]. Thus the author shows that Einstein's and Whitehead's Relativity Theories yield different results since Einstein's theory does not, in fact, give an acceleration of the center of gravity. *L. Infeld* (Warsaw).

Brahmachary, R. L. Sur les propriétés d'un modèle instable de cosmologie contenant un "fluide imparfait". Naturwissenschaften **40**, 456–457 (1953).

The author studies the stability of a static cosmological model containing an imperfect fluid. Comparisons are given with the behavior of an Einstein universe. *A. E. Schild.*

MECHANICS

Gomes, Ruy Luís. The true meaning of the principle of invariance of modern physics. Gaz. Mat., Lisboa **14**, no. 55, 1–3 (1953). (Portuguese)

Considerations de nature épistémologique sur la notion d'invariance en mécanique classique, relativité restreinte et relativité générale. L'auteur semble distinguer des coordonnées "mathématiques" et des coordonnées "données de signification physique", distinction qui ne parait pas très claire au rapporteur. *A. Lichnerowicz* (Paris).

Freudenstein, Ferdinand. An analytical approach to the design of four-link mechanisms. Trans. A.S.M.E. **76**, 483–489; discussion, 489–492 (1954).

The variable angles, angular velocities and accelerations in a four-bar linkage are expressed in terms of a new parameter P defined by $P^2=a^2+b^2+c^2+d^2-e^2-f^2$, where a, b, c, d are the constant lengths of the links and e, f are the variable lengths of the diagonals. Cyclic permutations of a, b, c, d give valid corresponding relations. Some of the derivations are contained in the discussion of the paper. Applications of the analysis of the linkages to various types of mechanisms are considered. *M. Goldberg.*

Froda, Alexandru. Le caractère des discontinuités des champs de forces dans la mécanique des mouvements réalisables. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **3** (1951), 435–440 (1952). (Romanian. Russian and French summaries)

That the notations and definitions are taken from previous papers of the author not available to the reviewer makes it impossible to analyse this note with certainty. The author draws some further consequences from a set of axioms of particle mechanics which he says he has published in Acad. Repub. Pop. Române Stud. Cerc. Mat. 3, no. 1 (1952). In so far as the reviewer can see, the author generalizes the classical equations by allowing the velocity and acceleration to fail to exist, provided the position field satisfies assumptions of "finitude" not clearly stated in this paper. He concludes that a steady force-field must be multi-valued at each point where the acceleration is discontinuous. It seems to the

reviewer that the author defines force as mass times generalized acceleration, and that hence this work concerns only the theory of ordinary differential equations, not mechanics.

C. Truesdell (Bloomington, Ind.).

Zeuli, Tino. Problemi relativi al moto di un punto su una sfera riferita a coordinate ellittiche sferiche. Boll. Un. Mat. Ital. (3) 9, 50–54 (1954).

The author formulates some of the fundamental equations relating to the motion of a particle on a sphere in terms of elliptic spherical coordinates. A particular case, in which the particle is acted on by the force derived from a potential point function U, and in which the equations of motion can be solved by quadratures, is discussed briefly.

L. A. MacColl (New York, N. Y.).

Backes, F. Sur quelques problèmes de dynamique. Mathesis 63, 5–9 (1954).

En faisant usage des théorèmes géomètriques simples, l'auteur démontre d'une manière élégante que l'orbite du mouvement képlérien et celle du mouvement d'un point attiré par un centre en raison directe de la distance sont des coniques.

O. Bottema (Delft).

Proca, A. Mécanique du point. J. Phys. Radium (8) 15, 65–72 (1954).

Let ξ^* be the spinor conjugate to ξ and γ^α the matrices for which $\gamma^{(\alpha}\gamma^{\beta)}=\delta^{\alpha\beta}$ $(\alpha,\beta=1,\cdots,4)$. The author postulates that the motion of a particle can be described by the spinors ξ and ξ^* as functions of a parameter τ, such that $dx^\alpha/d\tau=\xi^*\gamma^\alpha\xi$. A second postulate is that the Lagrangian for a free particle is given by

$$L=\frac{1}{2}\left(\frac{d\xi^*}{d\tau}\xi-\xi^*\frac{d\xi}{d\tau}\right)+\lambda_\alpha(\dot{x}^\alpha-\xi^*\gamma^\alpha\xi).$$

The spin is represented by the quantity $m_{\alpha\beta}=-\xi^*\gamma_{[\alpha}\gamma_{\beta]}\xi$. The velocity of the particle in the absence of a force field turns out to be the superposition of a constant velocity and an oscillation due to the spin. For the motion of a charged particle in an electromagnetic field the theory leads to the classical expression of the Lorentz force.

J. Haantjes.

Galliasot, F. Les formes extérieures en mécanique. Ann. Inst. Fourier Grenoble 4 (1952), 145–297 (1954).

Les caractéristiques de la forme extérieure $\omega=\omega_1+\omega_2$, défini par $\omega_1=mk_{ij}dv^i\wedge dn^j-mk_{ij}v^idv^j\wedge dt$; $\omega_2=k_{ij}X^idn^j\wedge dt$ (k_{ij} symbole de Kronecker) sur une variété V_7, sont les équations du mouvement d'un point matériel soumis à une force X^i. Il est montré qu'à tout système paramétrique holonome à n degrés de liberté est associé une forme Ω de degré 2 de rang $2n$, dont les caractéristiques sont les équations du mouvement. On peut se librer de la servitude de coordonnées en utilisant l'élément E de l'espace tangent à la variété V_{2n+1} pour lequel $i(E)\Omega=0$ (i est l'opérateur de l'antidérivation). Dans cette théorie une liaison imposée à un système se compose d'une relation $a(p_i,q^i,t)=0$ et une force pour réaliser cette liaison. Cette force se traduit par un champ de liaison F avec $i(E+F)da=0$. Il est montré qu'on peut déterminer des équations du mouvements d'un système de p liaisons comme caractéristiques d'une forme Ω de rang $2(n-p)$ jointes à p formes de Pfaff. L'auteur étudie la classe de liaisons où le champ F est de la forme λe (e connu à priori). Dans le cas où des signes sont imposés à priori aux facteurs λ et aux formes da (par exemple un solide en contact avec un plan) la question se pose de déterminer le mouvement ultérieur. La condition nécessaire et suffisante d'unicité des mouvements est déduite. Enfin il est montré que l'étude des systèmes différentiels comme caractéristiques d'une forme Ω s'effectue au moyen d'endomorphismes qui conduisent aux opérateurs antidérivation et dérivation de H. Cartan.

J. Haantjes (Leiden).

Sapa, V. A. On the motion of a material point on the surface of a rotating rough inclined cylinder. Izvestiya Akad. Nauk Kazah. SSR 1952, no. 116, Ser. Astr. Fiz. Mat. Meh. 1(6), 151–170 (1952). (Russian. Kazak summary)

The cylinder is right and circular, its axis is fixed, and the rotation uniform. The equations of relative motion are written for the case when friction exists even though the particle is not in contact with the surface. Fortunately, in processing the equations, the particle is assumed to be on the surface. Unfortunately, the friction is assumed to have the components Nk_1 along, and Nk_2 across the generator, where N is the normal reaction, and k_1, k_2 are constants. Since the trajectory is not assumed to be helical, friction does not have a direction opposite to that of the velocity, and is not of the conventional kind. Fortunately, the paper concentrates on the cases when one of the k's is negligible. If it is k_1, one equation becomes that of a pendulum with dry friction (quadratic damping). Unfortunately, when k_2 is negligible, it becomes difficult to find the questions to the paper's somewhat elaborate answers.

A. W. Wundheiler.

Haimovici, Adolf. Contributions à la mécanique du point de masse variable. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 4, 61–68 (1952). (Romanian. Russian and French summaries)

The author considers the motion of a particle of variable mass subject to external forces. The mass depends not only on the time but on a second parameter φ, such as temperature, which is defined at each point of the space. He takes as coordinates the orthogonal trajectories to the surfaces $\varphi=u_3=$const. and two other families of curves u_1 and u_2 on these surfaces. The equations of motion are formulated in terms of u_1, u_2 and u_3 and special cases are considered such as movement orthogonal to the surfaces $\varphi=u_3$ and movement in these surfaces. An application to the case of almost circular planetary motion is given when the surfaces $\varphi=u_3$ are concentric spheres.

R. A. Rankin (Birmingham).

Hydrodynamics, Aerodynamics, Acoustics

Zarantonello, Eduardo H. Parallel cavity flows past a plate. J. Math. Pures Appl. (9) 33, 29–80 (1954).

It is known that an obstacle placed in a uniform stream of an ideal fluid does not determine a unique steady irrotational flow if the fluid is allowed to cavitate. There are two aspects of this indeterminacy: a number of topologically different flow configurations can occur, and for a given configuration more than one flow may be possible. The author sets for himself the problem of finding all the configurations which are possible when the flow is plane and the obstacle is a flat plate. The main assumptions made are that the flow not involve a second sheet of the flow plane, and that there be no infinite velocities. These conditions unfortunately exclude at the outset several attractive and useful configurations (the reentrant jet, for example, is disallowed

for the former reason, and the finite trailing cavity for the latter). In conclusion the author finds that under the conditions laid down there are only nine configurations; in seven the plate is parallel to the stream and circulation is present, the eighth is the standard Kirchhoff flow with the plate arbitrarily inclined, and the last is the Kirchhoff flow with an added cavity in front of the plate. *J. B. Serrin.*

Yamada, Hikoji. On the slow motion of viscous liquid past a circular cylinder. Rep. Res. Inst. Appl. Mech. Kyushu Univ. **3**, no. 9, 11–23 (1954).

The two-dimensional problem which determines the flow field of a viscous liquid past a circular cylinder placed perpendicular to the uniform flow has been treated elegantly by S. Tomotika and T. Aoi by means of Oseen's linearized equations. The author treats the same problem by the same equations but in somewhat different way, with the view of determining the Reynolds number at which the rear twin-vortices make their appearance. The value found is about 3.02 which is not far from the experimental one 2.65, but this coincidence is rather contingent and Oseen's approximation in general seems to be remote from the exact solution of the Navier-Stokes equations except in the case of extremely small Reynolds number. This is indicated in the following by consideration of the pressure. (Author's summary.) *Y. H. Kuo* (Ithaca, N. Y.).

Kawaguti, Mitutosi. Numerical solution of the Navier-Stokes equations for the flow around a circular cylinder at Reynolds number 40. J. Phys. Soc. Japan **9**, 303 (1954).

Okabe, Jun-ichi. An approximate calculation of the flow of a viscous fluid past a body having a flat end. Rep. Res. Inst. Appl. Mech. Kyushu Univ. **3**, no. 9, 45–50 (1954).

The flow of a viscous incompressible fluid over, and behind, a semi-infinite rectangular body is calculated by the "modified" Oseen approximation. In the limiting case of a flat plate, the velocity distribution along the axis of the wake has been compared with the calculation of Goldstein [Proc. Roy. Soc. London. Ser. A. **142**, 545–560 (1933)]. As expected, the discrepancy is large. *Y. H. Kuo.*

Uberoi, Mahinder S. Correlations involving pressure fluctuations in homogeneous turbulence. NACA Tech. Note no. **3116**, 61 pp. (1954).

The basic formula of the paper is

$$(1)\qquad \frac{1}{\rho}\overline{pq'}(\boldsymbol{\xi})=\frac{1}{4\pi}\int\frac{\partial^2}{\partial y_i\partial y_j}\overline{u_iu_jq'}(\mathbf{y})\frac{d\mathbf{y}}{|\mathbf{y}-\boldsymbol{\xi}|}$$

where $\overline{pq'}(\boldsymbol{\xi})$ is the covariance of the pressure at one point in homogeneous turbulence with the value of some other physical quantity q at another point, a vector distance $\boldsymbol{\xi}$ away from it. The density (assumed constant) is ρ, and u_i is the velocity component in the direction of the ith coordinate axis. Particular cases of the formula have been used before [Batchelor, Proc. Cambridge Philos. Soc. **47**, 359–374 (1951); these Rev. **12**, 874; Limber, Proc. Nat. Acad. Sci. U. S. A. **37**, 230–233 (1951); these Rev. **12**, 874] but the author's results are fuller; they were obtained independently.

The formula yields simple results only if a further assumption is made, namely that fourth cumulants of the joint velocity distribution at different points can be neglected. Then any mean product $\overline{abcd}$ of four quantities can be replaced by a simpler expression,

$$(2)\qquad \overline{ab}\,\overline{cd}+\overline{ac}\,\overline{bd}+\overline{ad}\,\overline{bc}.$$

Some experimental confirmation of the accuracy of this hypothesis was given by Batchelor, who applied it to the case $a=b=c=d=u'-u$ (a velocity difference between neighbouring points). Measurements by the present author confirm this work, which shows that the formula in this case is accurate except for points very close together, when it is an underestimate (though by not more than 20%). The author goes on to give a far fuller experimental investigation, taking a, b, c and d to be various different velocity components at one or other of two neighbouring points. The error is in all these cases very much smaller, even for points very close together. This is not incompatible with the earlier result because that really involved a comparison of differences of means.

The author concludes that quantities like $\overline{pq'}$ can be estimated satisfactorily from the above equations. He gives various interesting special cases of this. One point, deducible already from Batchelor's and Limber's papers but not mentioned explicitly, seems especially interesting to the reviewer. The correlation of the pressure with the square of the fluid speed is about -0.05. Thus it is negative, as in steady flow (governed by Bernoulli's equation), but is reduced from 100% to 5%. *M. J. Lighthill.*

Lighthill, M. J. On sound generated aerodynamically. II. Turbulence as a source of sound. Proc. Roy. Soc. London. Ser. A. **222**, 1–32 (1954).

The theory of sound generated aerodynamically by isotropic turbulence developed in an earlier paper [same Proc. **211**, 564–587 (1952); these Rev. **13**, 879] is now extended to allow for the effects of internal motions ("convection") in the medium of appreciable Mach number (>0.3) and for shear flows. The author shows that the effect of a convective velocity with a Mach number $\mathbf{M}_c$ (assumed solenoidal) is to replace the integral

$$(1)\qquad \int\frac{(x_i-y_i)(x_j-y_j)}{|\mathbf{x}-\mathbf{y}|^3}\frac{\partial^2}{\partial t^2}T_{ij}\left(\mathbf{y},t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)d\mathbf{y},$$

(where a_0 is the velocity of sound and

$$T_{ij}=\rho v_iv_j+(p_{ij}-a_0^2\rho\delta_{ij})$$

represents the distribution of the quadrupole sources of sound) of the earlier theory into

$$(2)\qquad \int\frac{(x_i-y_i)(x_j-y_j)}{|\mathbf{x}-\mathbf{y}|^3}\left(1-\frac{\mathbf{M}_c\cdot(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right)^{-3}\frac{\partial^2}{\partial t^2}S_{ij}\times\left(\mathbf{n},t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)d\mathbf{n},$$

where $T_{ij}(\mathbf{y},t)=S_{ij}(\mathbf{n},t)$ and $\mathbf{n}$ is the Lagrangian variable associated with the time $t-|\mathbf{x}-\mathbf{y}|/a_0$. With suitable approximations (2) yields the formula

$$(3)\qquad i(\mathbf{x})\sim\frac{x_ix_jx_kx_l}{16\pi^2\rho_0a_0^5(x-\mathbf{M}_c\cdot\mathbf{x})^6}\times\int\left[\frac{\partial^2}{\partial t^2}T_{ij}(0,t)\frac{\partial^2}{\partial t^2}T_{kl}(\mathbf{z},t)\right]_{\mathrm{av}}d\mathbf{z},$$

for the intensity of the sound field at a point $\mathbf{x}$ far from the origin per unit volume of turbulence situated at the origin.

Similarly, when the term $p\partial\bar{v}_1/\partial x_2 = p\bar{e}_{12}$ (e_{kl} is the rate of strain tensor) due to shear in T_{ij} dominates, (3) becomes

$$i(\mathbf{x}) \sim \frac{x_1 x_2 \bar{e}_{12}(0)}{8\pi^2 \rho_0 a_0^5 (x - M_c x_1)^6} \int x_k x_l \bar{e}_{kl}(\mathbf{z}) \left[\frac{\partial p(0,t)}{\partial t} \frac{\partial p(\mathbf{z},t)}{\partial t} \right]_{\mathrm{av}} d\mathbf{z}.$$

This is approximated to

$$\frac{1+5M_c^2}{(1-M_c^2)^4} \frac{1}{15\pi\rho_0 a_0^5} \int\int \tau(\mathbf{y})\tau(\mathbf{z}) \left[\frac{\partial p(\mathbf{y},t)}{\partial t} \frac{\partial p(\mathbf{z},t)}{\partial t} \right]_{\mathrm{av}} d\mathbf{y} d\mathbf{z},$$

where $\tau(\mathbf{y})$ and $\tau(\mathbf{z})$ denote the components of mean shear in the direction of the shear at the origin. From this expression the result

$$\Pi_s = \frac{1+5M_c^2}{(1-M_c^2)^4} \frac{U^2 (\partial p/\partial t)^2_{\mathrm{av}} S}{15\pi\rho_0 a_0^5},$$

for the power output per unit area of turbulent shear layer is obtained; here U is the change in velocity across the layer,

$$S = \int \frac{\{[\partial p(\mathbf{y},t)/\partial t][\partial p(\mathbf{z},t)/\partial t]\}_{\mathrm{av}}}{[\partial p(\mathbf{y},t)/\partial t]^2_{\mathrm{av}}} dS_z,$$

and dS_z is an element of shear layer in $\mathbf{z}$-space. Detailed comparisons are made of the foregoing formulae with the existing experimental data on sound produced by jets and shear flows. *S. Chandrasekhar* (Williams Bay, Wis.).

✶Özoklav, Hasan. Sikiştirilabilen bir akişkanin cidarlari hiperbol şeklinde olan bir kanaldaki hareketine ait çözüm. [On the motion of a compressible fluid in a hyperbola-shaped channel.] Thesis, Istanbul Technical University, 1954. Kutulmuş Basimevi, Istanbul, 1954. 31 pp. (French summary)

The paper is concerned with the two-dimensional steady motion of a compressible inviscid nonconducting gas in a channel with hyperbolic walls. The author's starting point is the following unpublished result of R. Berker. If in a plane flow the stream lines and their orthogonal trajectories form an isometric net, the scalar velocity V has the following form: $V = g(\beta)G(\alpha, \beta)$, where α and β are orthogonal curvilinear coordinates ($\beta = ct$ corresponding to stream lines), $g(\beta)$ is an arbitrary function, and $G(\alpha, \beta)$ satisfies a second-order hyperbolic differential equation which is determined completely if the stream lines are given. When the stream lines are confocal hyperbolas, the author is able to transform this equation to an Euler-Poisson equation the solutions of which can be written in definite integral form. If the scalar velocity distribution is arbitrarily prescribed at the entrance and on the wall of the channel, the velocity distribution inside the channel can be obtained by making use of the above solutions. In the last chapter the velocity distribution is determined numerically for a particular case.

It should be noted that a velocity field constructed in the manner described by the author is valid only for a fictitious gas whose equation of state is determined by the boundary conditions imposed on the velocities. Therefore if the motion of a real gas is sought the velocities cannot be prescribed arbitrarily. *E. T. Onat* (Ankara).

Chester, W. The propagation of shock waves in a channel of non-uniform width. Quart. J. Mech. Appl. Math. 6, 440–452 (1953).

The problem considered is that of a plane shock wave of arbitrary strength propagating along a two-dimensional channel made up of two uniform sections of widths $2b$ and $2c$, respectively, separated by a non-uniform transition section of finite length, where the non-uniform transition occurs in such a manner that the problem can be linearized on the basis of small changes in the width of the channel. The particular case in which the transition region is of uniformly increasing width is treated by the use of the results of M. J. Lighthill [Proc. Roy. Soc. London. Ser. A. 198, 454–470 (1949); these Rev. 11, 478] on the diffraction of a shock wave around a corner (of angle $\pi+\delta$, where δ is small) and the method of images. The general case follows by superposition. It is found that at large distances from the transition region the final effect of any transition section is a change in shock strength corresponding to a pressure decrease behind the shock of $K(p_1-p_0)(c/b-1)$ where p_1-p_0 is the initial pressure jump across the shock and K is a parameter which decreases monotonically with shock strength ($0.5 \geq K \geq 0.394$). Similar results are found for the sound wave which, depending on whether the flow behind the shock is subsonic or supersonic, is propagated upstream or downstream, respectively. If the flow behind the shock is sonic, the pressure in the sound wave becomes infinite, indicating that for this case the small changes in channel width might possibly produce secondary shocks.
P. Chiarulli (Providence, R. I.).

Jones, C. W. On gas flow in one dimension following a normal shock of variable strength. Proc. Roy. Soc. London. Ser. A. 221, 257–267 (1954).

The problem of the non-uniform propagation of a normal shock wave and of the non-homentropic one-dimensional flow behind it is considered with the use of the equations of motion in Lagrangian form, entropy and time as independent variables. A particular family of exact solutions is developed and it is indicated how the actual solution, essentially dependent on a single first-order differential equation of a standard type, may in principle be found by quadratures. A particular member of the family corresponding to a shock advancing with a velocity which varies as the $-n/(n+2)$ power of the time into a region of gas initially at rest with a uniform pressure and a density varying as the nth power of the distance downstream ($n > -2$) is studied for strong shocks and it is shown that similar results apply for shocks of arbitrary strength. The case $n=1$ is studied in detail. The extension to spherically symmetric shocks is shown to be straightforward. *P. Chiarulli.*

Hida, Kinzo. An approximate study on the detached shock wave in front of a circular cylinder and a sphere. J. Phys. Soc. Japan 8, 740–745 (1953).

The author extends some work of K. Tamada (unpublished) and T. Kawamura [Mem. Coll. Sci. Univ. Kyoto. Ser. A. 26, 207–230 (1950)] on the detached shock wave in front of a circular cylinder or sphere moving supersonically through a perfect gas. The previous authors fitted an approximate incompressible potential flow for the subsonic flow between the nose of the shock wave and the nose of the body, while the present author fits an approximate rotational incompressible flow. A very approximate solution to this flow is obtained and the results differ very little from those corresponding to the irrotational flow.
P. Chiarulli (Providence, R. I.).

Cabannes, Henri. Influence des accélérations sur la courbure des ondes de choc. I. Ecoulements de révolution. C. R. Acad. Sci. Paris 238, 321–323 (1954).

The author shows that if a cone is placed in accelerating axisymmetrical flow at a supersonic speed fast enough for the nose shock wave to be attached, then the curvature of

that shock wave at the nose is proportional to the acceleration. The coefficient multiplying the acceleration is a function of nose semi-angle and upstream Mach number. In this very brief exposition the author has not been able to find space for an explicit expression for this function. For bodies of revolution of more general shape (asymptotically conical tip, but finite curvature of body section immediately behind it) the curvature of the shock wave is made up of the part, mentioned above, proportional to the acceleration of the body, and the steady-flow value (proportional to the curvature of the body section). This part was calculated previously by the author [Recherche Aéronautique no. **24**, 17–23 (1951); these Rev. **13**, 597].

It occurred to the reviewer that the author's result, if fully worked out, could be very useful in studying photographs like Mair's [Philos. Mag. (7) **43**, 695–716 (1952)] of flows in which nearly conical dead-air regions pass very rapidly up and down a long thin pole in front of a bluff body of revolution. The theory might make it possible to estimate the acceleration of the separation point (which forms the apex of the "cone") by observing the curvature of the boundary of the dead-air region. *M. J. Lighthill.*

Cabannes, Henri. Influence des accélérations sur la courbure des ondes de chocs. II. Ecoulements plans. C. R. Acad. Sci. Paris **238**, 448–449 (1954).

The paper deals with the problem exactly analogous to that of the preceding paper in the two-dimensional case. In this case, however, the coefficients are all given explicitly. *M. J. Lighthill* (Manchester).

Taub, A. H. Curved shocks in pseudo-stationary flows. Ann. of Math. (2) **58**, 501–527 (1953).

The methods and results of T. Y. Thomas [J. Math. Physics **26**, 62–68 (1947); **27**, 279–297; **28**, 62–90, 91–98, 153–172 (1949); these Rev. **8**, 611; **10**, 494, 758; **11**, 278, 479] pertaining to "shock-wave consistency relations" for stationary flows of an ideal gas are extended to cover the case of pseudo-stationary flows. It is found that the general form of the consistency relations (the nth derivative with respect to arc length along a streamline of the curvature of the streamline at a shock in terms of the parameters characterizing the shock and the first n derivatives with respect to arc length along the shock of the curvature of the shock, $n=0, 1, 2, \cdots$) is unchanged. In particular, for $n=0$, it is shown that the ratio of the curvature of the shock to the curvature of a pseudo-streamline at the shock is the same function of the parameters characterizing the shock for pseudo-stationary flow as it is for stationary flows and that the relation between the curvature of a pseudo-streamline and the pressure gradient is also unchanged.

An analysis is made of "allowable discontinuities" along a pseudo-stationary shock and of the nature of boundary curves separating regions of uniform flow and non-uniform flow. These results are used in an exhaustive analysis of the shock configuration arising from the regular reflection of a plane shock from a plane rigid wall of finite extent. *P. Chiarulli* (Providence, R. I.).

Pai, S. I. On the flow behind an axially symmetrical attached curved shock. J. Franklin Inst. **257**, 383–398 (1954).

In this paper the author extends his perturbation method for treating the flow behind a shock wave [see J. Aeronaut. Sci. **19**, 734–742 (1952); these Rev. **14**, 329] to the case of a curved shock attached to a body of revolution. The body is assumed to be approximately conical, so that the usual Taylor-Maccoll solution can be taken as the basic unperturbed flow. Depending on the free-stream speed and the angular opening of the cone, the Taylor-Maccoll flow will be subsonic, mixed, or supersonic, and correspondingly there are three cases to consider. In each case a computation scheme and appropriate formulas are given for determining the first perturbation of the stream function and the shock wave. *J. B. Serrin* (Cambridge, Mass.).

Legendre, Robert. Ecoulement supersonique autour d'un corps élancé. C. R. Acad. Sci. Paris **238**, 1863–1865 (1954).

Nocilla, Silvio. Su di un problema di aerodinamica relativo alle ali a delta. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15**, 177–186 (1953).

The author considers a delta wing with dihedral, in supersonic flow with its leading edges inside the Mach cone, and investigates the lift distributions due to small angles of incidence and yaw respectively. These are calculated in the form of series in powers of the dihedral angle; two terms of such a series are found for each lift distribution, so that if all four terms are used the expression neglects only the fourth power of the dihedral angle (apart from any errors involved in the use of the familiar linearized theory). *M. J. Lighthill* (Manchester).

Mirels, Harold. Aerodynamics of slender wings and wing-body combinations having swept trailing edges. NACA Tech. Note no. **3105**, ii+96 pp. (1954).

The author presents a comprehensive theory of the flow around slender wings which is based on the ideas (now classical) of Munk and R. T. Jones. The theory applies to the general case of a ring with swept-back trailing edges which may be attached to a central body. An important feature of the analysis is the introduction of generating functions which are derivatives of the complex velocity in the transverse plane. For a given wing configuration, the analysis leads to an integral equation. This can be solved explicitly for certain limiting cases, but in several others numerical methods must be resorted to. Wings in steady motion, or in roll, or in pitch, are considered and the resultant aerodynamic forces are calculated. A case of unsteady forward motion is also discussed (Wagner effect). It is pointed out that the analysis for swallow-tail wings given in a paper by the reviewer [Aeronaut. Quart. **4**, 69–82 (1952); these Rev. **14**, 219] is incomplete in the general case, although it is applicable to the particular case solved explicitly in that paper (i.e., the case of small sweepback). *A. Robinson* (Toronto, Ont.).

Martin, John C., and Gerber, Nathan. The effect of thickness on pitching airfoils at supersonic speeds. J. Math. Physics **33**, 46–56 (1954).

A two-dimensional aerofoil of thickness-ratio ϵ moving at uniform supersonic speed and constant rate of pitch q is investigated mathematically. The equation of motion is written down in a frame of reference rotating with angular velocity q, in which the aerofoil is at rest. This equation is solved by successive approximations, the first being ordinary linear theory, in which ϵ, q and the angle of attack α are assumed infinitesimal. Of the second approximation to the pressure field, the authors evaluate the terms in $q\epsilon$, and so find the effect of thickness on the additional pressure distribution due to pitch. Its effect on the relevant derivatives

(lift due to pitch, pitching moment due to pitch) is then determined by integration. The results are displayed graphically for a 5% thick symmetrical parabolic arc section, and compared with the results of a linear theory.

In the analysis, difficulties due to the discontinuity in flow direction at the leading edge are got over by a technique suggested to the authors by work of Van Dyke. The discontinuity is smoothed out over a finite distance, which is permitted to vanish only at the end of the investigation. It has been verified that this procedure gives the correct answer in iteration solutions of non-linear problems in which the solution was already known, so it may be valuable in many more in the future. (Note that in non-linear problems ordinary delta-function procedure is not available.)

M. J. Lighthill (Manchester).

Kestin, K., and Zaremba, S. K. Adiabatic one-dimensional flow of a perfect gas through a rotating tube of uniform cross section. Aeronaut. Quart. 4, 373–399 (1954).

The problem treated is that encountered in a jet propelled helicopter in which the rotor blades are driven by jets fed by uniform channels in the rotor blades. The centrifugal force due to the rotation modifies the ordinary equations of one-dimensional flow; the authors develop the corresponding modified theory. They encounter a nonlinear differential equation of the first order. They discuss its singularities and are thereby able to predict the behavior of the gas flow in the channel qualitatively. For the angular velocity of rotor rotation below a certain critical value, the flow is always subsonic at an interior point of the pipe, but, for sufficiently high flow velocity at the entrance it may become sonic at the outer end. For rotor angular velocity higher than this critical value, the same phenomena may occur except that the channel may have an interior region of supersonic flow. Quantitative results are only mentioned except that the equations are integrated approximately when the flow velocity is small.

E. Pinney (Berkeley, Calif.).

Chambré, P. L. Speed of a plane wave in a gross mixture. J. Acoust. Soc. Amer. 26, 329–331 (1954).

The author shows that to derive the speed of sound in a gross mixture it is not necessary to assume a relation between the compressibility of the mixture and those of the components; that relation follows from the relation between the densities of the three. He derives the speed of sound independently and shows that it agrees with those obtained with the use of the additional assumption by A. B. Wood [A textbook of sound, Bell, London, 1930, pp. 326–329] and R. J. Urick [J. Appl. Phys. 18, 983–987 (1947)].

J. Shmoys (New York, N. Y.).

Voit, S. S. Passage of spherical sound waves from a moving medium into a medium moving with another speed and having different properties. Doklady Akad. Nauk SSSR (N.S.) 92, 491–493 (1953). (Russian)

This extends a previous paper of the author [Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 157–164 (1953); these Rev. 14, 1142] on a point source at the point $(0, 0, h)$, the media in $z>0$ and $z<0$ being homogeneous and distinct. Both media now have uniform velocities, not necessarily parallel, whereas previously one was at rest.

F. V. Atkinson.

Miles, John W. On nonspecular reflection at a rough surface. J. Acoust. Soc. Amer. 26, 191–199 (1954).

A plane scalar wave, of propagation constant k, is to strike the approximately plane interface $z=\zeta(x, y)$ between two infinite media, where $|k\zeta|\ll 1$, $|\zeta_x|\ll 1$, $|\zeta_y|\ll 1$. If ζ^2 and higher powers are neglected in the boundary conditions, the problem admits an explicit solution, of which the author considers particularly special cases. For a rigid interface with two- or three-dimensional roughness the first-order perturbation term ψ in the reflected wave is found in general by the stationary phase method, subject to further restrictions on ζ; for sinusoidal roughness a direct method gives ψ as possibly exponentially attenuated, the interface thus acting, so far as non-specular reflection is concerned, as a band-pass wave filter. The method is also applied to the cases of a pulse and a rigid interface, and of an elastic interface, either sinusoidal or arbitrary. Previous research is carefully surveyed.

F. V. Atkinson (Ibadan).

***Levine, Harold. Acoustic radiation pressure on a circular disk.** Proceedings of Symposia in Applied Mathematics, Vol. V, Wave motion and vibration theory, pp. 63–69. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. $7.00.

L. V. King's solution [Proc. Roy. Soc. London. Ser. A. 153, 1–16 (1935)] of the title problem for normal incidence of a plane, monochromatic wave is improved by a variational formulation, the details of which follow closely an earlier analysis of the scattering by a fixed disk [H. Levine and J. Schwinger, Physical Rev. (2) 74, 958–974 (1948); these Rev. 10, 221]. King's results, applicable only at low frequencies, are extended to include approximations valid at high frequencies.

J. W. Miles (Los Angeles, Calif.).

Elasticity

Klyušnikov, V. D. Derivation of the Beltrami-Michell equations from the variational equations of Castigliano. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 250–252 (1954). (Russian)

The author derives the Beltrami-Michell equations from a variational principle of the form

$$\delta\int_\Omega\bigg\{V+\lambda_1\left(\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}+\rho X\right)+\lambda_2\left(\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}+\rho Y\right)+\lambda_3\left(\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}+\rho Z\right)\bigg\}d\Omega=0,$$

where V is the strain energy per unit volume, expressed in terms of the stresses, (X, Y, Z) is the extraneous force per unit mass, $X_x, X_y, \cdots$ are the stresses, and $\lambda_1, \lambda_2, \lambda_3$ are Lagrangian multipliers.

J. L. Ericksen.

Schumann, Walter. Sur différentes formes du principe de B. de Saint-Venant. C. R. Acad. Sci. Paris 238, 988–990 (1954).

This note adds to the wealth of existing confusion regarding Saint-Venant's principle in the linear theory of elasticity. It seems appropriate to clarify certain historical statements appearing in the paper before commenting on what is new in its content.

The author refers to Boussinesq's version of the principle [Application des potentiels . . . , Gauthier-Villars, Paris, 1885, p. 296] and subsequently cites von Mises' formulation

[Bull. Amer. Math. Soc. **51**, 555–562 (1945); these Rev. **7**, 40], which he calls a precise statement of Boussinesq's version. He then states that the "general proof" of the principle, in the form given by von Mises, is due to R. Wick [Dissertation, Technische Hochschule, München, 1950]. Finally, he refers to a paper [Sternberg, Quart. Appl. Math. **11**, 393–402 (1954); these Rev. **15**, 370] which contains a general proof of the principle as conjectured by von Mises, as a "new proof" of the principle "in the form given by Boussinesq".

The following comments are in order. (a) As pointed out by von Mises, Boussinesq's formulation of the principle is deficient in meaning and thus incapable of proof. (b) As shown by von Mises with the aid of counter-examples, Boussinesq's version, when properly clarified, is false. (c) Von Mises' formulation is an essential modification, and not merely a more rigorous interpretation, of Boussinesq's statement. (d) The reviewer has seen a copy of Wick's dissertation, which is unpublished. It does not contain a general proof of the principle in the von Mises formulation. Instead, it presents a rough intuitive argument which amounts merely to an abridged version of von Mises' treatment of the special example of the half-space.

The original contribution of the paper consists of four theorems which are asserted without proof. The statement of these theorems is not clear to the reviewer if the author uses the symbol "O", for order of magnitude, in the conventional sense. *E. Sternberg* (Chicago, Ill.).

✱v. Krzywoblocki, M. Z. On the socalled principle of least work. Proceedings of The First Midwestern Conference on Solid Mechanics, April, 1953, pp. 43–48. The Engineering Experiment Station, University of Illinois, Urbana, Ill. 1954.

Babuška, Ivo. The plane problem of elasticity. Časopis Pěst. Mat. **77**, 227–240 (1952). (Czech)

This is an expository article on the plane problem of the classical (linear) mathematical theory of elasticity, with emphasis on modern function-theoretic and integral-equation methods, and especially on the work of the Russian school. The presentation is in the main descriptive and brief, without proofs or references, but within these limitations it is precise, and emphasises conditions of validity.

A. Erdélyi (Pasadena, Calif.).

Wegner, U. Zwei Probleme der ebenen Elastizitätstheorie. Z. Angew. Math. Mech. **33**, 300–303 (1953).

The following two problems are solved as variational problems. I. A rectangular plate with a coaxial rectangular hole is rotating about one of its axes and one wants to find the strains at the boundary points of the axis. A very good approximation of the solution is given. More details are given in the (unpublished) thesis of Fr. Henn [Darmstadt]. II. Two identical rectangular plates are fastened together along one of their equal sides. These two plates are heated differently and one asks for the distribution of strains on their common side. *C. Arf* (Istanbul).

Ionescu-Cazimir, Viorica. Sur les équations de l'équilibre thermoélastique. IV. Le cas plan. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 547–554 (1952). (Romanian. Russian and French summaries)

For plane stress subject to the classical linear theory of thermoelasticity, the author observes that the displacement components and the Airy function satisfy $\Delta^2(\Delta-\nu\partial/\partial t)\phi=0$, where Δ is the Laplacian operator and ν is a material constant, while the temperature and the dilatation satisfy $\Delta(\Delta-\nu\partial/\partial t)\phi=0$. Hence she obtains a general solution in terms of three potentials satisfying one or the other of these equations. Also she constructs two analytic functions of $x+iy$ from linear combinations of dilatation ϑ and rotation ω; one of these is $(\lambda+2\mu)\vartheta-\beta\omega+\mu i\omega$, generalizing the classical result to which it reduces when the thermoelasticity β is zero. *C. Truesdell* (Bloomington, Ind.).

Herzig, Alfred. Zur Torsion von Stäben. Z. Angew. Math. Mech. **33**, 410–428 (1953). (English, French and Russian summaries)

The author gives all details of solving the torsion problem of a half-circle and quarter-circle cross-section. No reference is made to previous solution of these and extended problems by Ghosh [Bull. Calcutta Math. Soc. **39**, 107–112 (1947); **40**, 107–115 (1948); these Rev. **10**, 84, 496], Mitra [ibid. **42**, 131–144 (1950); these Rev. **12**, 559] and other writers. The maximum values of the boundary shear-stress are determined. A final problem is the full circular section with a radial slit. The same value of the torsional stiffness has been given in Drang und Zwang [Bd. II, 2. Aufl., Oldenbourg, München-Berlin, 1928, §74] by Föppl for this problem. It is 55.9 percent of a solid circular shaft.

D. L. Holl (Ames, Iowa).

✱Reissner, Eric. On finite torsion of cylindrical shells. Proceedings of The First Midwestern Conference on Solid Mechanics, April, 1953, pp. 49–51. The Engineering Experiment Station, University of Illinois, Urbana, Ill., 1954.

The author solves the problem of torsion of any cylindrical shell which possesses a plane of symmetry perpendicular to the generators of the cylinder, using the equations of Marguerre [Proc. 5th Internat. Congress Appl. Mech., Cambridge, Mass., 1938, Wiley, New York, 1939, pp. 93–101]. He derives a torque-twist relation from which he computes the maximum torque which the tube can sustain and shows that, if the torque-twist relation is approximated by a polynomial of degree three in the twist, the maximum torque calculated from the approximate form differs considerably from that calculated previously. Since approximations similar to this are made in obtaining the basic equations, it seems to this reviewer that one cannot place much confidence in the former calculation of the maximum torque.

J. L. Ericksen (Washington, D. C.).

Agababyan, E. H. Dynamic expansion of an elastic cylinder. Ukrain. Mat. Žurnal **5**, 375–379 (1953). (Russian)

An elastic cylinder, $1<r<b/a$, is suddenly subject to uniform and constant internal pressure for time $t>0$. Stresses σ_r and σ_θ are found graphically, proceeding from the characteristics $dr=\pm dt$. *R. E. Gaskell* (Seattle, Wash.).

Teodorescu, P. P. Sur le théorie exacte de l'équilibre des surfaces cylindriques. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 111–194 (1952). (Romanian. Russian and French summaries)

The author considers the Love theory of cylindrical shells in the formulation for anisotropic materials given by Girkmann [Flächentragwerke, Springer, Vienna, 1946, pp. 323–331, 348–422]. His objective is to obtain a general solution which can be adapted to fit boundary conditions. This work contains probably the most extensive application

† In my review I expressed lack of confidence in one of the author's results. He has since convinced me that it is extremely unlikely that his result ~~should~~ could be shown ~~that~~ not to be a good approximation in the situations to which it is intended to apply.

of the matrix method of Gr. C. Moisil yet presented. [This method has been illustrated in many recent Romanian papers, most of which refer to a book by Moisil, Matricele asociate sistemelor de ecuaţii cu derivate parţiale, Bucarest, 1950, which the reviewer has not been able to see.] The author shows (¶9) that all three displacements satisfy a single homogeneous linear partial differential equation of eighth order with constant coefficients, these latter being very elaborate functions of the elastic moduli of the material (Appendices 1 and 2). Next (¶11) he shows that the displacements may be expressed as linear combinations of fifth derivatives of a single function satisfying the above equation, the coefficients again being elaborate functions of the elastic moduli (Appendices 3 and 4). Various associated differential equations are discussed in §§3–4. Most of the rest of the paper (§§5–7) concerns the simplifications resulting in various special cases or approximations. These special cases are not individual problems but rather classes of problems defined by the vanishing of certain resultants: the extensional ("membrane") theory, Finsterwalder's theory, Dischinger's theory, etc. In §8 the author discusses the numerical magnitudes of some of the coefficients for anisotropic bodies. A great deal of labor has gone into this work. *C. Truesdell* (Bloomington, Ind.).

Smith, R. C. T. The bending of a semi-infinite strip. Australian J. Sci. Research. Ser. A. 5, 227–237 (1952).

The author solves the problem of the bending of a thin semi-infinite strip in the case that the long edges are clamped and the short edge is subjected to arbitrary displacements and couples. This amounts to solving the biharmonic equation $\nabla^4 w=0$ in a half-strip under the boundary conditions: $w=\partial w/\partial n=0$ on the edges of infinite length, and $w=f$ and $\nabla^2 w=g$ are prescribed functions on the short edge.

Using the technique of separation of variables, the author reduces the problem to a non-self-adjoint boundary-value problem for a system of two second-order ordinary differential equations. The coefficients are constants, and the interval is finite (being the short edge of the strip). The associated eigenfunction expansion is then written down in rather explicit fashion. The eigenvalues are the roots of certain transcendental equations, and they have been elsewhere tabulated. The mathematical problem is to validate the expansion. The author succeeds in this by applying the method of contour integration, as used by Titchmarsh, to the non-self-adjoint matrix differential operators in question. *J. Berkowitz* (New York, N. Y.).

Müller, W. Zur Theorie der Vierpilzplatte. Ing.-Arch. 22, 60–72 (1954).

This paper concerns the linear bending theory of isotropic plates under transverse load. Solutions are given of the problems involving four symmetrically situated and equal concentrated forces when two opposite edges of the plate are simply supported and the other two edges are either simply supported or are free from shear force and have zero slope. Results are generally presented in closed form through the use of Theta functions. Numerical results are given for special cases. The application is to the design of mushroom floors. *H. G. Hopkins* (Sevenoaks).

Nowacki, Witold. Beitrag zur Theorie der orthotropen Platten. Acta Tech. Acad. Sci. Hungar. 8, 109–128 (1954). (Russian, English and French summaries)

This paper concerns the linear bending theory of orthotropic plates under transverse load. First the Green's function is found in closed form for an infinitely-long rectangular plate simply-supported along its edges. Second a Green's function approach is employed in the solution of problems for a rectangular plate, simply-supported along three edges and partly built-in and partly simply-supported along the remaining edge. *H. G. Hopkins* (Sevenoaks).

***Truesdell, C. A new chapter in the theory of the elastica.** Proceedings of the First Midwestern Conference on Solid Mechanics, April, 1953, pp. 52–55. The Engineering Experiment Station, University of Illinois, Urbana, Ill., 1954.

Define a free-shape problem to be that of determining the shape which must be given to an elastic body in order that it will assume an assigned shape when loaded in a specified manner. As the author points out, this is a new class of boundary-value problems in non-linear elasticity, but not in the fully linear theory. For the elastica, this problem can be solved by quadratures. The author illustrates this by exhibiting the quadrature for the case of a plane cantilever loaded normally by a uniform load and obtains a bound on the error made in the nth of a series of successive approximations of the integral. Detailed results for a beam which is straight when loaded are given. *J. L. Ericksen* (Washington, D. C.).

***Mindlin, R. D. Force at a point in the interior of a semi-infinite solid.** Proceedings of The First Midwestern Conference on Solid Mechanics, April, 1953, pp. 56–59. The Engineering Experiment Station, University of Illinois, Urbana, Ill., 1954.

The problem referred to in the title (Mindlin's problem) was first solved by the author in 1935 [C. R. Acad. Sci. Paris 201, 536–537 (1935); Physics 7, 195–202 (1936)]. The original solution was obtained by superposition upon Kelvin's solution, corresponding to a force at a point of a medium occupying the entire space, of solutions of the field equations which possess singularities outside a half-space containing the point of application of the force, and which annul the tractions on the plane boundary of the half-space.

In the present paper the solution is reached by more direct and systematic means. The author first extends his completeness proof [Bull. Amer. Math. Soc. 42, 373–376 (1936)] for the solution of the field equations appropriate to a linear, isotropic, elastic medium, in terms of the Boussinesq-Papkovich stress functions, to the case in which body forces are present. He then considers the problem of the half-space subjected to body forces which vanish outside a bounded region T surrounding the load-point P. By means of the Boussinesq-Papkovich stress functions this problem is reduced to the problem in potential theory which consists in determining a function from its given boundary values and the known value of its Laplacian throughout the region under consideration. Thus an integral representation of the solution, involving Green's function of the first kind, is established. The solution to the concentrated-load problem is obtained in closed form through an elementary limit process in which T is shrunk to P while the resultant body force is made to tend to the prescribed concentrated load. *E. Sternberg* (Chicago, Ill.).

Matumoto, Tosimatu. Transmission and reflection of seismic waves through multilayered elastic medium. Bull. Earthquake Res. Inst. Tokyo **31**, 261–273 (1953). (Japanese summary)

Building on the matrix method of Thomson [J. Appl. Phys. **21**, 89–93 (1950); these Rev. **11**, 702] and Torikai [J. Acoust. Soc. Japan **8**, 21–27 (1952)] the author sets up the boundary conditions for longitudinal and transverse seismic waves in an elastic medium with n parallel planes of discontinuity of elastic-wave velocity and solves the matrix equations (1) when the multilayered medium exists between two semi-infinite elastic media; (2) when the multilayered medium is bounded by a free surface on one side and a semi-infinite elastic medium on the other. A numerical solution is given for the case when $n=2$, that is when there is but one layer between two semi-infinities (1) when it is of lower velocity and (2) when it is of higher velocity. The calculated amplitude ratios are tabulated and graphed as functions of the ratio of layer thickness to wave length. It is a paper of much interest to seismologists.

J. B. Macelwane (St. Louis, Mo.).

Suhara, Toshiro. A study on the damping of longitudinal vibration of an elastic cylinder. Rep. Res. Inst. Elasticity Eng. Kyushu Univ. **7**, 25–38 (1951).

This paper discusses the vibrations of an elastic rod in the form of a right circular cylinder, one end of which is attached to the plane surface of an elastic half-space, the other end being free. The main aspect investigated is the damping caused by propagation of waves into the elastic foundation. The solution in the rod takes the form of an expansion in the radius r which is considered small compared with the length. The solution in the foundation is in accordance with the theory of elasticity. The Laplace transform is used, and the numerical displacement at the joint is expressed in the form of a complicated contour integral. For numerical evaluation a simplified approximation is used for the displacement in the rod which is independent of the radius, and simple expressions for the damping constants are obtained. Specific numerical examples are cited.

E. H. Lee (Providence, R. I.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Deryugin, L. N. On the theory of diffraction from a reflecting grid. Doklady Akad. Nauk SSSR (N.S.) **93**, 1003–1006 (1953). (Russian)

The author develops further his analysis of the problem of a plane wave incident on a perfectly conducting plane with parallel rectangular grooves [same Doklady (N.S.) **87**, 913–916 (1952); these Rev. **15**, 183], assuming now that the width of a groove is small compared to the incident wave-length. After various approximations he finds the field components at the opening to a groove in terms of two functions, for which graphs are given. Special cases of the groove-depth are discussed. It appears that the analysis does not apply when the incident wave-length approximates to a submultiple of the period of the grid. *F. V. Atkinson.*

Rice, S. O. Diffraction of plane radio waves by a parabolic cylinder. Calculation of shadows behind hills. Bell System Tech. J. **33**, 417–504 (1954).

The problem is that of diffraction of a plane electromagnetic wave by a perfectly conducting parabolic cylinder. The author starts from the ordinary expansion of the field into parabolic cylinder functions of integral order. Then he transforms the series into a contour integral and evaluates the contour integral after shifting the path of integration. The general procedure is essentially the same as that used by Watson [Proc. Roy. Soc. London. Ser. A. **95**, 83–99 (1918)] in the case of a spherical obstacle. The results are also very similar to those obtained in the case of diffraction by a sphere when the characteristic dimension of the surface is large compared to wavelength. The author also compares the solution in the case of a parabola with vanishingly small latus rectum to that obtained in the limiting case of the half-plane.

Both the far-field and the surface-currents distribution are examined in detail, and the relation between them is studied. The parabolic cylinder functions which naturally appear in the problem are discussed in detail and their asymptotic properties, necessary to obtain a good representation of the solution ought, are investigated. The results obtained are applied to the problem of propagation of radio waves in the presence of hills. *J. Shmoys.*

Stevenson, A. F. Solution of electromagnetic scattering problems as power series in the ratio (dimension of scatterer)/wavelength. J. Appl. Phys. **24**, 1134–1142 (1953).

The fields inside and outside the scattering body are expressed formally as power series in the ratio mentioned in the title. The calculation of successive terms requires the solution of standard problems of potential theory, together with the evaluation of certain potential integrals, so that the process can be carried out as far as desired if Laplace's equation is soluble in the coordinate system appropriate for the body. The work of the author is a fine extension of Rayleigh's, and clarifies certain difficulties concerning the match between the near and distant fields. Some features of similar nature in a paper by Tai [Trans. I. R. E. **PGAP-1**, 13–36 (1952)] are criticized. *C. J. Bouwkamp.*

Stevenson, A. F. Electromagnetic scattering by an ellipsoid in the third approximation. J. Appl. Phys. **24**, 1143–1151 (1953).

Author's abstract: The method previously described [see preceding review] . . . is here applied to the ellipsoid, the first three terms in the series being obtained. The second term in the series for the wave-zone field (that proportional to k^3) vanishes, and the same is true for any body possessing a center of symmetry. Thus the terms in k^2, k^4 in the wave-zone field are here obtained. The direction and polarization of the incident wave, and the electromagnetic constants of the ellipsoid, are arbitrary. The final results are expressed in terms of certain elliptic integrals which are functions of the three principal axes of the ellipsoid. These integrals can all be expressed simply in terms of just two such integrals; they become elementary integrals in the case of a spheroid. Various special cases are considered, including that of a perfectly conducting elliptical disk and the complementary problem of diffraction through an elliptical hole in a perfectly conducting screen. *C. J. Bouwkamp.*

Twersky, V. Reflection coefficients for certain rough surfaces. J. Appl. Phys. **24**, 659–660 (1953).

Verf. hat die Reflexion elektromagnetischer Wellen durch vollkommen leitende Oberflächen berechnet, welche entweder aus Halbzylindern oder aus Halbkugeln bestehen, welche auf einer Ebene aufgesetzt sind [dasselbe J. **22**, 825–835 (1951); diese Rev. **13**, 305]. Im Vorliegenden behandelt er diese Aufgabe wieder, indem er von der einfallenden Wellenleistung ausgeht und diese vergleicht mit der von einer Einheitsfläche beim Spiegelwinkel reflektierten Leistung. Bei der Ausarbeitung dieses Gedankens geht Verf. von dem Fall der Halbzylinder aus. Die analytische Durchführung ergibt Grenzen für die Gültigkeit des Ergebnisses, welches mit dem früheren identisch ist. *M. J. O. Strutt.*

Quantum Mechanics

Reichenbach, Hans. Les fondements logiques de la mécanique des quanta. Ann. Inst. H. Poincaré **13**, 109–158 (1953).

The author begins with a discussion of the role of indeterminism in classical and quantum physics, based on the use of the probability concept. Among other things, he indicates the difficulty of reconciling the formalisms of quantum statistics with the point of view that the quantum theory provides a statistical description of a more exact and detailed theory. He then deals with the description of observed and unobserved objects in the quantum theory and shows that a three-valued logic is appropriate for an interpretation of quantum physics which is free from what he refers to as "causal anomalies". This is followed by a discussion of the order and the direction of time in classical physics and the relation of the direction of time to the change of entropy. Finally the author discusses the order and the direction of time in quantum physics. Among other things he points out that the direction of time is not determined by the Schroedinger equation and that the possible interpretation of the positron as an electron with its world line directed toward the past [E. C. G. Stueckelberg, Helvetica Phys. Acta **14**, 588–594 (1941); **15**, 23–37 (1942); these Rev. **4**, 56; R. P. Feynman, Physical Rev. (2) **76**, 749–759 (1949)] leads to a lack of uniqueness in the order of time in quantum physics. *N. Rosen* (Haifa).

Proca, Alexandre. Quantification en mécanique spinorielle. C. R. Acad. Sci. Paris **238**, 774–776 (1954).

The author requires that the components of the basic four-spinor of his recent Spinor Mechanics [J. Phys. Radium (8) **15**, 65–72 (1954); these Rev. **15**, 836] be operators. Formal consequences of requiring that these operators satisfy commutation or anticommutation conditions are developed. Comparison with experience eliminates the possibility that they satisfy anticommutation relations. *A. J. Coleman* (Toronto, Ont.).

Rayski, Jerzy. On the energy of bound states in quantum field theory. I. Acta Phys. Polonica **13**, 51–65 (1954). (Russian summary)

The total Hamiltonian for the case of charged, spinless, non-relativistic particles, interacting through the electromagnetic field, is considered for the problem of the eigenvalues of the energy. The decisive step is the implicit solution of the equation $\Box A_\mu = -j_\mu$, in the form

$$A_\mu(x) = A_\mu{}^0(x) + A_\mu{}^{st}(x) - \int D^{t_0}(x, x')\ddot{A}_\mu{}^{st}(x')d^4x.$$

Here $\Box A_\mu{}^0 = 0$, $A_\mu{}^{st}$ is a solution (in particular,

$$A_\mu{}^{st}(\mathbf{x}, t) = (4\pi)^{-1}\int j_\mu(\mathbf{x}', t)(|\mathbf{x}-\mathbf{x}'|)^{-1}d^3x')$$

of $\Delta A_\mu{}^{st}(\mathbf{x}, t) = -j_\mu(\mathbf{x}, t)$, and

$$D^{t_0}(x, x') = \begin{cases} -D(x-x') & t > t' > t_0, \\ +D(x-x') & t_0 > t' > t, \\ 0 & \text{elsewhere,} \end{cases}$$

$D(x)$ being the Jordan-Pauli function. $D^{t_0}(x, x')$ vanishes, together with the time-derivative for $t = t_0$, and satisfies $\Box D^{t_0}(x, x') = -\delta(x - x')$. This solution for A_μ is introduced into the total Hamiltonian. The current j_μ, however, itself depends on A_μ. An iteration procedure then gives the Hamiltonian as a power series in the coupling constant, this being evaluated up to the second order. The energy operator splits into several parts; the kinetic energy of charged fields, the energy of photons, the interaction energy between photons and the charged fields, several self-energy terms, and cross-terms which in configuration space give the Coulomb interaction, Breit terms and some other corrections. *A. Salam* (Cambridge, England).

Klein, Abraham. The Tamm-Dancoff formalism and the symmetric pseudoscalar theory of nuclear forces. Physical Rev. (2) **90**, 1101–1115 (1953).

Some of the methods and results of Lévy's papers on the non-adiabatic treatment of the relativistic two-body problem [Physical Rev. (2) **88**, 72–82, 725–739 (1952); these Rev. **14**, 706] are re-examined with the primary intention to add cogency and completeness. Lévy generalized the T.D. (Tamm-Dancoff) equal-times formalism, and showed that the results obtained were equivalent to those obtained from the R.E. (relativistic two-body equation of Bethe and Salpeter). The procedure is now reversed; the T.D. formalism as generalized by Lévy is deduced from the R.E. The essence of the procedure is the establishment of a connection between a single covariant interaction term of given order of the R.E. (expressed by an appropriate four-dimensional integral and represented by a single Feynman diagram), and the number and kinds of the more conventional matrix elements to which this term reduces in the more conventional T.D. formalism. This connection provides a convenient guide for enumerating all matrix elements of a specified type and precludes the possibility of omission of any members of the set. Rules are given for writing down any matrix element. The formalism is applied to the derivation of static nuclear potentials. The fourth-order adiabatic potential is computed for the symmetric pseudoscalar theory. Greater accuracy than in Lévy's papers is used in the expansion of the energy denominators associated either with intermediate states containing a nucleon pair, or with states in which there are no pairs and one or more mesons. This brings to light: (1) contributions cancelling with all other two-pair matrix elements that are of relative order μ/M (μ meson, M nucleon mass); (2) one-pair terms which do not vanish but yield a repulsive interaction appreciably more significant at small distances than the second-order central potential. The derivation of six- and eighth-order potentials is outlined. The leading terms in the many-particle problem are analyzed, and the results are applied to three- and four-body forces. *E. Gora* (Providence, R. I.).